On the rationalization of the $K(n)$-local sphere

Notes by the participants for a reading seminar at MPIM

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0 The isomorphism between the Lubin-Tate tower and the Drinfeld tower (Peter Scholze, 27 May)

We begin with something classical: the *modular curve*

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^{\pm},$$

where $\mathbb{H}^\pm$ denotes both the upper and lower half planes: that is, $\mathbb{H}^\pm := \mathbb{P}^1_{\mathbb{C}} \setminus \mathbb{P}^1(\mathbb{R})$. In fact, these are the $\mathbb{C}$-points of an algebraic curve $\mathcal{M}$ over $\mathbb{Q}$. To see this, we identify it as the moduli of elliptic curves:

Theorem 0.1 (Uniformisation Theorem; Riemann)**.** *There is a bijection between:*

1. *Elliptic curves over $\mathbb{C}$,*
2. *Pairs (Λ, W), where Λ is a free $\mathbb{Z}$-module of rank 2 and $W \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ is sub-$\mathbb{C}$-vector space of rank 1 such that $W \cap \overline{W} = 0$.*

Concretely, the equivalence is given as follows: given any elliptic curve E, there is an exponential

$$\exp : \operatorname{Lie} E \to E$$

which exhibits E as the quotient of $\operatorname{Lie} E \cong \mathbb{C}$ by a lattice $\Lambda \subset \mathbb{C}$. Recording this quotient is equivalent to remembering the kernel W of $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} \twoheadrightarrow \mathbb{C}$; conversely, any pair (Λ, W) defines an elliptic curve $E := \mathbb{C}/\Lambda$, provided that W satisfies the condition $W \cap \overline{W} = 0$ that ensures we quotient by a nondegenerate lattice. Fixing the $\mathbb{Z}$-module Λ, this recovers $\mathcal{M}(\mathbb{C})$ as the quotient $\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm$: the choice of W gives a point in $\mathbb{P}^1_{\mathbb{C}}$, with the condition $W \cap \overline{W}$ cutting out the complement of $\mathbb{P}^1(\mathbb{R})$. In particular, this exhibits $\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm$ as an algebraic curve over $\mathbb{Q}$ (in fact, $\mathbb{Z}$). More generally, there is a similar classification for complex tori. Moreover, there is a tower of coverings of $\mathcal{M}(\mathbb{C})$ where we instead take quotients of $\mathbb{H}$ by congruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$.

Question. Is there a similar 'p-adic uniformization'

$$(\mathcal{M} \otimes \mathbb{C}_p)^{\mathrm{ad}} \cong {}_{\Gamma}\backslash^{\mathbb{H}_p},$$

as the quotient of a 'p-adic symmetric space' $\mathbb{H}_p$ by the action of some arithmetic group Γ?

Answer. In this generality, no. However, there are many related results:

- one variant will give rise to the Lubin-Tate tower for GL_2 (or more generally, GL_h),
- another variant will give rise to the Drinfeld tower.

0.1 The Lubin-Tate tower

Claim. There exists an open subset of $(\mathcal{M}\otimes\mathbb{C}_p)^{\mathrm{ad}}$ that does admit such a p-adic uniformisation.

To see this, let's look first at $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{F}_p$. This has two strata: an open stratum $\mathcal{M}^{\mathrm{ord}}$ of ordinary elliptic curves, and a closed stratum $\mathcal{M}^{\mathrm{ss}}$ of supersingular elliptic curves. Here an elliptic curve E is called *supersingular* if its endomorphism algebra (over $\overline{\mathbb{F}}_p$) is of rank 4; equivalently, the torsion subgroup $E[p^\infty]$ is infinitesimal, or again equivalently, $E[p]$ has no nonzero points over a field.

Remark 0.2. The term 'supersingular' is something of a misnomer, as these are still nonsingular curves. It originates from considering 'singular' values of the j-invariant.

When E is supersingular, its endomorphism algebra is given by $\mathcal{O}_D$, the maximal order in a division algebra $D/\mathbb{Q}$ with

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_v = \begin{cases} M_2(\mathbb{Q}_v), & v \neq p, \infty; \\ \text{nonsplit}, & v = p, \infty. \end{cases}$$

We will give a uniformisation of the rigid-analytic open subset $\mathcal{U} \subset (\mathcal{M} \otimes \mathbb{C}_p)^{\mathrm{ad}}$ of all points having supersingular reduction from $\mathbb{C}_p$ to $\overline{\mathbb{F}}_p$.

Remark 0.3. Being supersingular can be phrased as a vanishing condition over $\overline{\mathbb{F}}_p$; before passing to the residue field this boils down to a certain function on $\mathbb{C}_p$ taking value < 1; in particular an open condition, so $\mathcal{U}$ really is open.

Theorem 0.4. *As rigid-analytic spaces, we have*

$$\mathcal{U} \cong \coprod \mathcal{O}_D^{\times} \backslash \mathring{\mathbb{D}}_{\mathbb{C}_p},$$

where $\mathring{\mathbb{D}}_{\mathbb{C}_p}$ denotes the open unit disc.

Remark 0.5. The coproduct accounts for the (finite) choice of supersingular elliptic curve over $\overline{\mathbb{F}}_p$.

The open unit disc that shows up is the generic fibre of the Lubin-Tate space, and the quotient by $\mathcal{O}_D^{\times}$ accounts for the automorphisms of a supersingular elliptic curve. As in the complex setting, this moduli space is in fact the zero-th level in a tower of spaces encoding level structures.

To prove the theorem, we will discuss a more canonical definition of Lubin-Tate space. For this, we start with a commutative, one-dimensional formal group $G/\overline{\mathbb{F}}_p$ of height h; in fact, we will pick G so that it is already defined over $\mathbb{F}_p$.

Example 0.6. At height 2, we can take $G = E[p^\infty] = \widehat{E}$, where E is a supersingular elliptic curve.

Warning 0.7. At height $h > 2$, it is no longer the case that we can form G as the completion of an abelian variety.

Serre-Tate theory shows that deforming E (over some p-adic ring) is equivalent to deforming $E[p^\infty]$; the latter is in general much more manageable.

Definition 0.8. The *Lubin-Tate space* $\mathcal{M}_{\mathrm{LT},0}$ is the deformation space of G. In other words, $\mathcal{M}_{\mathrm{LT},0}$ assigns to any Artin local ring R with residue field $\overline{\mathbb{F}}_p$ the set of $\star$-isomorphism classes of deformations of G to R.

Since $\mathcal{M}_{\mathrm{LT},0}$ is smooth and its tangent space is understandable, we obtain the following identification:

Theorem 0.9 (Grothendieck, Illusie)**.** $\mathcal{M}_{\mathrm{LT},0} \cong \operatorname{Spf} W(\overline{\mathbb{F}}_p)[\![u_1, \dots, u_{n-1}]\!]$.

Remark 0.10. The coordinates $u_1, \dots, u_{h-1}$ are not canonical, but u_i is well-defined up to $(p, u_1, \dots, u_{i-1})$.

In particular, we obtain a universal deformation G^{univ} over $W(\overline{\mathbb{F}}_p)[\![u_1, \dots, u_{n-1}]\!]$, satisfying

$$[p]_{G^{\mathrm{univ}}}(X) = X^{p^h} + u_{h-1}X^{p^{h-1}} + \dots + u_1 X^p + p.$$

The choice of coordinates presents the (geometric) rigid analytic generic fibre $\mathcal{M}_{\mathrm{LT},0,\mathbb{C}_p} \cong \mathring{\mathbb{D}}_{\mathbb{C}_p}$ as the $(h-1)$-dimensional open unit disc. Analogous to the congruence tower in the complex case, we have a tower

$$\mathcal{M}_{\mathrm{LT},m,\mathbb{C}_p} \to \mathcal{M}_{\mathrm{LT},0,\mathbb{C}_p},$$

where $\mathcal{M}_{\mathrm{LT},m,\mathbb{C}_p}$ parameterises deformations (G', ε) together with a trivialisation $\tau \colon G'[p^m] \xrightarrow{\sim} (\mathbb{Z}/p^m\mathbb{Z})^h$ of the p^m-torsion.

Remark 0.11. Number theorists care about this tower because its cohomology is supposed to realise the local Langlands correspondence. A priori it's not so clear how to make sense of $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p} = \text{“}\lim\text{”}\,\mathcal{M}_{\mathrm{LT},m,\mathbb{C}_p}$, since arbitrary limits don't exist in adic spaces: the issue is that one would like to take a completed colimit of the algebras appearing at finite level, but this completion does not exist canonically. One can nevertheless make sense of its cohomology, as the colimit of the cohomologies at finite level. Weinstein was computing with this gadget and realised that $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$ *does* exist as a perfectoid space. More specfically, write $\mathcal{M}_{\mathrm{LT},m} = \operatorname{Spf} A_m$, and $\mathfrak{m} \subset A_0$ for the maximal ideal. Weinstein showed that the formal spectrum of $A := (\varinjlim A_m)^\wedge_{\mathfrak{m}}$ has the right universal property for maps out of perfectoid spaces [Wei16], [SW13, Prop. 2.4.5]. As a result, these days people usually think of $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$ as a *diamond* (more on that in later lectures!).

Returning to local Langlands: note that the group $\mathrm{GL}_h(\mathbb{Z}_p)$ acts on $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$, by the limit of the actions of $\mathrm{GL}_h(\mathbb{Z}/p^m\mathbb{Z}) = \mathrm{Aut}((\mathbb{Z}/p^m\mathbb{Z})^h)$. In fact, one can extend this to an action of the larger group $\mathrm{GL}_n(\mathbb{Q}_p) \times_{\det,\mathbb{Q}_p^\times} \mathbb{Z}_p^\times$. There is also an action of $\mathrm{Aut}(G) = \mathcal{O}_D^\times$, where now $D = D_{1/h}$ is the $\mathbb{Q}_p$-division algebra with $\mathrm{inv}(D) = 1/h$; for example, at height two D is the nonsplit quaternion algebra. Finally there is an action of the Weil group $W_{\mathbb{Q}_p} \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, coming from the fact that G was already defined over $\mathbb{F}_p$. These actions commute, and one has the following:

Theorem (Carayol's Conjecture; Deligne ($h = 2$), Harris-Taylor ($h > 2$))**.** *For $\ell \neq p$, one has an isomorphism*

$$H^*_{\text{ét}}(\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_\pi \pi \otimes \mathrm{JL}(\pi) \otimes \mathrm{LLC}(\pi)$$

as $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times \times W_{\mathbb{Q}_p}$-representations.

Let us clarify some of the terms in the theorem:

(IM) Check these are the right adjectives.

1. JL denotes the *Jacquet-Langlands correspondence*

$$\mathrm{JL} : \left\{ \begin{array}{c} \text{Irreducible discrete series} \\ \text{representations of } \mathrm{GL}_n(\mathbb{Q}_p) \end{array} \right\} \xrightarrow{\sim} \{\text{Irreducible representations of } D^\times\}.$$

2. LLC denotes the *local Langlands correspondence*

$$\mathrm{LLC} : \left\{ \begin{array}{c} \text{Irreducible admissible} \\ \text{representations of } \mathrm{GL}_n(\mathbb{Q}_p) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Irreducible } h\text{-dimensional Frobenius} \\ \text{semi-simple representations of } W_{\mathbb{Q}_p} \end{array} \right\}.$$

As such, the space $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$ exhibits simultaneously both the Jacquet-Langlands and the local Langlands corresepondence. The case $\ell = p$ is the subject of active research.

0.2 A second variant of p-adic uniformisation

An alternative is to replace the modular curve $\mathcal{M} = \mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm$ by a more general ‘Shimura curve’

$$\mathcal{M}' := \mathcal{O}_D^\times \backslash \mathbb{H}^\pm,$$

where now $D/\mathbb{Q}$ is a quaternion algebra which is *split* at ∞ and nonsplit at some finite place, say p. In particular, $\mathcal{O}_D^\times \hookrightarrow M_2(\mathbb{R})$ gives an action of $\mathcal{O}_D^\times$ on $\mathbb{H}^\pm$.

Theorem (Čerednik)**.** *There is an isomorphism of rigid-analytic spaces*

$$(\mathcal{M}' \otimes_{\mathbb{Q}} \mathbb{C}_p)^{\mathrm{ad}} \cong \Gamma \backslash (\mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p))$$

for a certain arithmetic group Γ described below.

The space $\mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$ appearing in the theorem is *Drinfeld's p-adic upper half-plane.* To describe the group Γ, we denote by D' the $\mathbb{Q}$-division algebra with

$$D' \otimes_{\mathbb{Q}} \mathbb{Q}_v = \begin{cases} M_2(\mathbb{Q}_p), & v = p; \\ \text{nonsplit}, & v = \infty; \\ D \otimes_{\mathbb{Q}} \mathbb{Q}_v, & v \neq p, \infty. \end{cases}$$

Then $\Gamma := \mathcal{O}_{D'}[1/p]^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q}_p)$, and in particular acts on $\mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$. Drinfeld's insight was that $\mathcal{M}'$ also arises from a moduli problem, this time classifying Serre's 'fake elliptic curves': these consist of an abelian surface A, together with an action of $\mathcal{O}_D$ on A.

Remark 0.12. The explanation for the existence of p-adic uniformisation in this case is that in $\mathcal{M}'$, all points have supersingular reduction at p. The moral is that in a general Shimura variety one can only uniformise over the 'basic locus', i.e. supersingular points. Here deformation theory of the abelian variety corresponds to deformation of its p-divisible group, again by Serre-Tate.

The theorem above was first proved by Čerednik using group-theoretic methods. The story is that Manin told Drinfeld to give a talk on this, and he subsequently discovered the following moduli-theoretic interpretation. One fixes $G_0/\overline{\mathbb{F}}_p$ a one-dimensional formal group of height h, so that $\mathrm{End}(G_0) = \mathcal{O}_{D_{1/h}}$. Then $\mathcal{O}_{D_{1/h}}$ also acts on $G := G_0^h$, and the *Drinfeld space at level zero* is defined to be the functor sending a p-power torsion $W(\overline{\mathbb{F}}_p)$-algebra to the set $\mathcal{M}_{\mathrm{Dr},0}(R)$ of triples (H, ρ, ε), where

- H is a formal group of dimension h,
- $\rho \colon \mathcal{O}_{D_{1/h}} \to \mathrm{End}(H)$,
- $\varepsilon \colon H \times_R R/p \sim G \times_{\overline{\mathbb{F}}_p} R/p$ is a quasi-isogeny of height zero.

Remark 0.13. We would like to take for $\mathcal{M}_{\mathrm{Dr},0}$ the deformation space of G (i.e., take ε to be an isomorphism), but the resulting space is not big enough.

Theorem (Drinfeld)**.** *The moduli problem $\mathcal{M}_{\mathrm{Dr},0}$ is representable by a p-adic formal scheme.*

Warning 0.14. The formal scheme that appears in the theorem is quite complicated: for example at height 2, each irreducible component of $\mathcal{M}_{\mathrm{Dr},0,\overline{\mathbb{F}}_p}$ is a $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$, arranged according to the Bruhat-Tits building of $\mathrm{GL}_2(\mathbb{Q}_p)$. A given component $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$ intersects the other components along $\mathbb{P}^1(\mathbb{F}_p)$.

In spite of this, the generic fibre is simple: for general h one has

$$\mathcal{M}_{\mathrm{Dr},0,\mathbb{C}_p} \cong \mathbb{P}^{n-1}_{\mathbb{C}_p} \setminus \bigcup_H \mathbb{P}(H),$$

where the union is over all $\mathbb{Q}_p$-rational hyperplanes. In particular, for $h = 2$ this gives

$$\mathcal{M}_{\mathrm{Dr},0,\mathbb{C}_p} \cong \mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p).$$

Once again, this is the zero-th level in a tower of $(\mathcal{O}_{D_{1/h}}/p^m)^\times$-torsors

$$\mathcal{M}_{\mathrm{Dr},m,\mathbb{C}_p} \to \mathcal{M}_{\mathrm{Dr},0,\mathbb{C}_p},$$

where $\mathcal{M}_{\mathrm{Dr},m,\mathbb{C}_p}$ parameterises deformations (H, ρ, ε) together with an isomorphism of $\mathcal{O}_{D_{1/h}}/p^m$-modules

$$H[p^m] \cong \mathcal{O}_{D_{1/h}}/p^m.$$

As in the Lubin-Tate case, we get commuting actions of $\mathcal{O}^\times_{D_{1/h}}$, $(\mathrm{GL}_n(\mathbb{Q}_p) \times_{\det,\mathbb{Q}_p^\times} \mathbb{Z}_p^\times)$ and $W_{\mathbb{Q}_p}$ on the cohomology $H^*(\mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p}, \overline{\mathbb{Q}}_p)$ of the limit. These are defined as follows:

- the $\mathcal{O}_{D_{1/n}}^\times$-action is the limit of the actions of $(\mathcal{O}_{D_{1/h}}/p^m)^\times$ on $\mathcal{M}_{\mathrm{Dr},m,\mathbb{C}_p}$,
- the $\mathrm{GL}_h(\mathbb{Q}_p) \times_{\mathbb{Q}_p^\times} \mathbb{Z}_p^\times$-action by coordinate changes on $G \cong G_0^h$,
- the $W_{\mathbb{Q}_p}$-action from the fact that G_0 was already defined over $\mathbb{Q}_p$.

This suggests the question of comparing the cohomologies of the two towers. Indeed we have:

Theorem (Faltings)**.** *There is an isomorphism*

$$\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p} \cong \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p},$$

equivariant for everything in sight.

Remark 0.15. 1. $\mathcal{M}_{\mathrm{LT}}$ and $\mathcal{M}_{\mathrm{Dr}}$ arise from very different deformation problems, so such an isomorphism should be seen as surprising.

2. When Faltings gave his proof of the theorem, the language did not yet exist to talk about $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$ and $\mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p}$ as geometric objects, so even making good sense of the statement of the theorem was hard work.

Example 0.16. At height 2, we get a diagram

$$\begin{tikzcd}
& \mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p} \cong \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p} \arrow[loop above, "\mathrm{GL}_2(\mathbb{Z}_p)\times\mathcal{O}_D^\times"] \arrow[dl, "\mathrm{GL}_2(\mathbb{Z}_p)"'] \arrow[dr, "\mathcal{O}_D^\times"] & \\
\mathring{\mathbb{D}}_{\mathbb{C}_p} & & \mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)
\end{tikzcd}$$

The bottom arrows are proétale torsors for the displayed group, and equivariant for the remaining action on each side.

In the rest of the lecture, we'll try to explain the proof of the isomorphism.

0.3 Comparing the two towers

In both cases we have constructed towers whose $\mathbb{C}_p$-points classify data of the following form:

- a p-divisible group $H/\mathcal{O}_{\mathbb{C}_p}$, possibly with an action;
- an isomorphism $H(\mathbb{C}_p) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h$, possibly equivariant for some action;
- an isogeny $H_{\mathcal{O}_{\mathbb{C}_p}/p} \sim G$, again maybe equivariant.

The key problem is therefore to understand the classification of p-divisible groups over $\mathcal{O}_{\mathbb{C}_p}$ and over $\overline{\mathbb{F}}_p$. The crucial and beautiful fact about p-divisible groups is that they can always be understood in terms of linear algebra, like the Riemann classification over $\mathbb{C}$. This might lead us to some fancy linear algebra, but at the end of the day it's just linear algebra. In close analogy to Riemann's classification, we have the following theorem in the p-adic setting:

Theorem ([SW13], Theorem B)**.** *There is an equivalence between the following categories:*

1. *p-divisible groups over $\mathcal{O}_{\mathbb{C}_p}$,*
2. *Pairs (T, W), where T is a finite free $\mathbb{Z}_p$-module and $W \subset T \otimes_{\mathbb{Z}_p} \mathbb{C}_p$ is a sub-$\mathbb{C}_p$-vector space.*

The correspondence sends a p-divisible group G to the Hodge-Tate filtration *on T_pG.*

On the other hand, the classification over the residue field is classical:

Theorem (Dieudonné)**.** *There is an equivalence between the following categories:*

1. *p-divisible groups over $\mathcal{O}_{\mathbb{C}_p}/p$,*
2. *Triples (M, F, V), where M is a finite free $W(\overline{\mathbb{F}}_p)$-module, $F\colon M \to M$ is* Frob*-linear and $V\colon M \to M$ is* Frob^{-1}*-linear.*

This begs the following question:

Question. Given a pair (T, W) corresponding to a p-divisible group $H_{T,W}/\mathcal{O}_{\mathbb{C}_p}$, what is the Dieudonné module of $H_{T,W} \times_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{O}_{\mathbb{C}_p}/p$? In other words, we want a concrete description of the arrow marked '?' below:

$$\begin{tikzcd}
\{p\text{-divisible groups}/\mathcal{O}_{\mathbb{C}_p}\} \arrow[r, "\sim"] \arrow[d, "/p"'] & \{(T, W)\} \arrow[d, "?"] \\
\{p\text{-divisible groups}/\overline{\mathbb{F}}_p\} \arrow[r, "\sim"'] & \{(M, F, V)\}
\end{tikzcd}$$

This requires a suitable formulation of p-adic Hodge theory: T is related to étale cohomology of the generic fibre of H, and M to crystalline cohomology of the special fibre. Note however that we need a form of p-adic Hodge theory over $\mathbb{C}_p$: for example, the infinite level space $\mathcal{M}_{\mathrm{LT},\infty}$ has no points over discretely valued fields (since it's perfectoid).

Constructing such a theory was one of the original motivations for the *Fargues-Fontaine curve*. This is a particular scheme $X_{\mathbb{C}_p}$, which is locally the spectrum of a PID. Its construction and properties will occupy a large part of the first half of the seminar. There is a point

$$\infty = \mathrm{Spec}(\mathbb{C}_p) \hookrightarrow X_{\mathbb{C}_p},$$

and indeed the residue fields at *all* closed points are complete algebraically closed fields—i.e., big. Indeed in one incarnation, $X_{\mathbb{C}_p}$ classifies untilts of $\mathbb{C}_p^\flat$: that is, pairs $(C, C^\flat \cong \mathbb{C}_p^\flat)$. By construction, vector bundles on $X_{\mathbb{C}_p}$ are closely related to isocrystals, i.e. rational Dieudonné modules. The following gives us a way to attack the question above:

Theorem ([SW13], Theorems A and C)**.** *The Dieudonné module functor*

$$\{p\text{-divisible groups over } \mathcal{O}_{\mathbb{C}_p}/p \text{ up to isogeny}\} \to \mathrm{VB}(X_{\mathbb{C}_p}),$$

is fully faithful, and its essential image consists of those vector bundles with slopes in $[0, 1]$.

Given this, we arrive at a description of '?'. Suppose given a pair (T, W), and form the following cartesian diagram:

$$\begin{tikzcd}
E(T, W) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & (i_\infty)_* W \arrow[d, hook] \\
T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathbb{C}_p}} \arrow[r] & (i_\infty)_*(T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathbb{C}_p}})
\end{tikzcd} \tag{0.17}$$

Note that $E(T, W)$ is a submodule of $T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathbb{C}_p}}$, and so torsion-free, and hence defines a vector bundle on $X_{\mathbb{C}_p}$.

Theorem (Scholze-Weinstein)**.** *If $G/\mathcal{O}_{\mathbb{C}_p}$ corresponds to the pair (T, W), then $G \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{O}_{\mathbb{C}_p}/p$ corresponds to $E(T, W)$.*

In fact, one can further describe isogenies between p-divisible groups over $\mathcal{O}_{\mathbb{C}_p}/p$ in terms of vector bundles on $X_{\mathbb{C}_p}$.

Corollary 0.18. *1. $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$ classifies the following equivalent data:*

   a) Tuples (T, W, α) , where $\alpha \colon T \cong \mathbb{Z}_p^h$ and $E(T, W) \cong \mathcal{O}_{X_{\mathbb{C}_p}}(-1/h)$,

   b) Inclusions $\mathcal{O}_{X_{\mathbb{C}_p}}(-1/h) \hookrightarrow \mathcal{O}^h_{X_{\mathbb{C}_p}}$ with cokernel supported at ∞.

2. $\mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p}$ classifies D-linear inclusions $\mathcal{O}_{X_{\mathbb{C}_p}}(-1/h)^h \hookrightarrow D \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathbb{C}_p}}$ with cokernel supported at ∞

Combining the above gives the following interpretation of Falting's theorem:

Theorem 0.19 ([SW13], Theorem E)**.** *There is a natural equivariant equivalence of adic spaces*

$$\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p} \simeq \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p}.$$

1 Power operations on Lubin-Tate theory (Nikolay Konovalov, 3 June)

1.1 The Morava Stabilizer group

Let us fix a prime p (once and for all) and a one-dimensional formal group Γ_h of height h over $\overline{\mathbb{F}}_p$.

Remark 1.1. Note that we are working over an algebrically closed field so that the choice of such a formal group of height h is unique up to isomorphism.

Definition 1.2. Define $\mathbb{G}_h$ to be the group $\mathrm{Aut}(\Gamma_h, \overline{\mathbb{F}}_p)$ consisting of pairs (f, g) where $f \colon \Gamma_h \xrightarrow{\sim} \Gamma_h$ is an automorphism of Γ_h that covers an automorphism $g \colon \overline{\mathbb{F}}_p \xrightarrow{\sim} \overline{\mathbb{F}}_p$. This profinite group is called the Morava stabilizer group.

Remark 1.3. Per construction, we have a short exact sequence

$$1 \to \mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_h) \to \mathbb{G}_h \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1 \tag{1.4}$$

where the latter arrow sends a pair (f, g) to the automorphism g. It is easy to see that this map admits a section $g \mapsto (\mathrm{id}, g)$ so that this short exact sequence splits.

In fact, one can identify the outside terms in this split short exact sequence as follows.

- The Galois group is given by $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$, the profinite completion of the integers,
- The automorphism group is given by $\mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_h) \cong \mathcal{O}_D^\times$, the the units in the ring of integers of a certain division algebra over $\mathbb{Q}_p$.

Note that central simple algebras over $\mathbb{Q}_p$ are classified by the Brauer group $\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$, and the division algebra D appearing above corresponds to the class $1/h$ in this Brauer group, and is of dimension h^2 over $\mathbb{Q}_p$. Taking the semidirect product of the factors above, we conclude that

$$\mathbb{G}_h \cong \widehat{D^\times},$$

i.e. the Morava stabiliser group is the profinite completion of the group of units in D.

Lemma 1.5 (cf. Remark 2.2.5 in [Mor85])**.** *If we let $\mathbb{G}_h$ act on the Witt vectors $W(\overline{\mathbb{F}}_p)$ by the Galois action, there is an identification of the rationalised continuous group cohomology as*

$$H^*_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^*_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, \ldots, x_h).$$

The right hand side is an exterior algebra on h generators where x_i is in degree $2i - 1$.

Proof. The first identification is immediate, as $\mathbb{G}_h$ is compact. Indeed, its semidirect factors are compact; this is immediate for $\widehat{\mathbb{Z}}$, and for $\mathcal{O}_D^\times$ we note that it morally should be like $\mathrm{GL}_h(\mathbb{Z}_p)$ as the group of units in the ring of integers in a division algebra of dimension h^2. Therefore, we can bring the filtered colimit computing rationalisation inside (continuous) group cohomology.

The second identification follows from the Lyndon–Hochschild–Serre spectral sequence associated to the short exact sequence (1.4). Since we are working with rational coefficients, this collapses and takes the form of an isomorphism

$$H^*_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cong H^*_{\mathrm{cont}}(\mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_h); \mathbb{Q}_p).$$

Now note that the automorphism group on the right hand side is isomorphic to $\mathcal{O}_D^\times$ hence is a p-adic analytic Lie group. A theorem of Lazard then states that its rational continuous group cohomology can be computed as Lie algebra cohomology as follows.

$$H^*_{\mathrm{cont}}(\mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_h); \mathbb{Q}_p) \cong \mathrm{H}^*_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{O}_D^\times); \mathbb{Q}_p).$$

Now the Lie algebra of the units in the ring of integers of D is just D itself with the commutator bracket, we can further identify this as

$$\mathrm{H}^*_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{O}_D^\times); \mathbb{Q}_p) \cong \mathrm{H}^*_{\mathrm{Lie}}(D; \mathbb{Q}_p).$$

Finally, for K sufficiently large over $\mathbb{Q}_p$, Morita theory tells us that we can identify the Lie algebra $D \otimes_{\mathbb{Q}_p} K$ with $\mathfrak{gl}_h(K)$. The Lie algebra cohomology of the latter is precisely the exterior algebra in the statement of the Lemma. □

Remark 1.6. From this computation, we see that $\mathbb{G}_h$ has finite virtual cohomological dimmension, in fact

$$\mathrm{vcd}_{\mathbb{Q}_p}(\mathbb{G}_h) = h^2 + 1.$$

This is also the strict cohomological dimension if $p - 1$ does not divide h.

1.2 The Lubin–Tate ring

Definition 1.7. A deformation of a formal group Γ/k is the datum of

- a complete local ring with residue field k, i.e. $(R, \mathfrak{m}_R, R/\mathfrak{m}_R \cong k)$,
- a formal group Γ_R over R, and
- an isomorphism $\Gamma_R \otimes_R k \cong \Gamma$.

Theorem 1.8 (Lubin–Tate)**.** *Consider the functor*

$$\mathrm{Def}_\Gamma \colon \mathrm{CRing}^{\mathrm{cpl,cts}} \to \mathrm{Set}$$

that sends a complete local ring with residue field k to the set of deformations of Γ to R. Then this functor is corepresented by a complete local ring $A(\Gamma, k)$ called the Lubin–Tate ring. In fact, there is a (non-canonical) presentation

$$A(\Gamma, k) \cong W(k)[\![u_1, \ldots, u_{h-1}]\!].$$

In the case where $(\Gamma, k) = (\Gamma_h, \overline{\mathbb{F}}_p)$ as above, we will simply write A for the associated Lubin–Tate ring.

Remark 1.9. Several observations can be made immediately from the universal property.

- The group $\mathbb{G}_h$ acts continuously on A.
- This action is horrendous. In particular, the augmentation map $A \to W(\overline{\mathbb{F}}_p)$ is not $\mathbb{G}_h$-equivariant for $h \geq 2$.
- However, the inclusion of constant terms $W(\overline{\mathbb{F}}_p) \to A$ is $\mathbb{G}_h$-equivariant.

We can now state the main theorem of the seminar.

Theorem 1.10. *Let $s \geq 0$. Then the inclusion map $W(\overline{\mathbb{F}}_p) \to A$ as above induces a split injection*

$$\phi\colon H^s_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \to H^s_{\mathrm{cont}}(\mathbb{G}_h; A).$$

Furthermore, the complement of this split injection is killed by some power of p depending on h and s. In particular, the induced map

$$H^s_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^s_{\mathrm{cont}}(\mathbb{G}_h; A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

Conjecture 1.11. It is conjectured that the final isomorphism above already holds before rationalisation.

Remark 1.12. The first part of the theorem, namely the fact that ϕ is a split injection, requires nontrivial topological methods including higher ambidexterity to provide transfers along surjective group homomorphisms.

1.3 Morava K-theories

This section largely follows Section 2 of [BB20b].

Definition 1.13. A unital associative ring spectrum A is said to be a division algebra if every A-module M splits as a direct sum of copies of A:

$$M \simeq \bigoplus_i \Sigma^{n_i} A.$$

Definition 1.14. Two division algebras A, B are of the same chromatic characteristic if and only if $A \otimes_{\mathbb{S}} B \neq 0$.

Remark 1.15. We can make the same definition in $\mathrm{Mod}_{\mathbb{Z}}$; in this case we see that two division algebras A, B in $\mathrm{Mod}_{\mathbb{Z}}$ are of the same (chromatic) characteristic if for any prime p we have that $p \cdot 1_A = 0$ if and only if $p \cdot 1_B = 0$. This does not mean that we have classified *all* division algebras in $\mathrm{Mod}_{\mathbb{Z}}$, but it is easy to find minimal representatives, i.e. the finite field $\mathbb{F}_p$ for any prime p and the rational numbers $\mathbb{Q}$.

Proposition 1.16 (Morava)**.** *Let k be a perfect field of characteristic p. Let Γ be a formal group of dimension one and height h (possibly infinite) over k. Then there exists a multiplicative cohomology theory $K(\Gamma, k)^*$ such that*

1. *The coefficients are given by*

$$K(\Gamma,k)^*(\mathrm{pt}) \cong \begin{cases} \mathbb{Q}, & h=0 \\ k[v_h^{\pm}], & 0<h<\infty \\ k, & h=\infty \end{cases}$$

   where v_h *is in degree* $2p^h-2$.

2. $K(\Gamma,k)^*$ *is complex orientable, and we can identify*

$$\mathrm{Spf}(K(\Gamma,k)^0(\mathbb{CP}^\infty)) \cong \Gamma.$$

   Note that the ring on the left hand side is given by $k[\![x]\!]$ *with the Hopf algebra structure arising from the Pontryagin comultiplication.*

3. $K(\Gamma,k)^*$ *satisfies a Künneth formula.*

Remark 1.17. By Brown representability, we see that $K(\Gamma,k)^*$ is represented by a unital associative ring spectrum $K(\Gamma,k)$.

Some basic examples are as follows:

- $K(\widehat{\mathbb{G}}_a,\mathbb{Q})$ and $K(\widehat{\mathbb{G}}_a,\mathbb{F}_p)$ are $\mathrm{H}\mathbb{Q}$ and $\mathrm{H}\mathbb{F}_p$ respectively.
- $K(\widehat{\mathbb{G}}_m;\mathbb{F}_p)$ is a summand of mod p complex K-theory.
- The standard Morava K-theories arise as $K(h,p) \coloneqq K(\Gamma_h,\mathbb{F}_p)$.

Theorem 1.18 (Devinatz–Hopkins–Smith)**.** *The Morava K-theories form a complete and pairwise distinct set of representatives for the chromatic characteristics of division algebras in spectra. Moreover, any division algebra is a module over some* $K(h,p)$.

Remark 1.19. For $0<h<\infty$, the Morava K-theories $K(h,p)$ do not admit the structure of $\mathbb{E}_2$-algebras in spectra.

Remark 1.20. The additional height variable in the minimal division algebras encodes specialisation: If X is a finite spectrum , then $K(h,p)^*X=0$ implies $K(h+1,p)^*X=0$. Therefore $K(h+1,p)$ can be thought of as a specialisation of $K(h,p)$.

1.4 Chromatic fracture

Let $\mathrm{Sp}_{(p)} \subset \mathrm{Sp}$ denote the full subcategory of p-local spectra. Since we now fix a prime p, we write $K(h) \coloneqq K(h,p)$. Recall that X is said to be $K(h)$-local if for all spectra Y such that $K(h)^*Y=0$, we have $[Y,X]=0$. The $K(h)$-local spectra span a full subcategory $\mathrm{Sp}_{K(h)} \subset \mathrm{Sp}_{(p)}$.

Theorem 1.21 (Bousfield)**.** *The inclusion* $\mathrm{Sp}_{K(h)} \subset \mathrm{Sp}_{(p)}$ *admits a left adjoint denoted* $L_{K(h)}$, *given by the Bousfield localisation with respect to* $K(h)$.

Similarly, define a functor

$$L_h\colon \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{(p)}$$

as the Bousfield localisation functor with respect to the spectrum $K(0)\oplus\cdots\oplus K(h)$.

Theorem 1.22 (Chromatic convergence, Hopkins–Ravenel)**.** *If $X \in \mathrm{Sp}^{\mathrm{fin}}_{(p)}$ is a p-local finite spectrum, then the map*

$$X \to \varprojlim(\cdots \to L_h X \to L_{h-1}X \to \cdots \to L_0 X)$$

is an equivalence

Theorem 1.23 (Smash product theorem, Hopkins–Ravenel)**.** *The functor $L_h\colon \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{(p)}$ commutes with colimits, hence is given by $L_h = L_h\mathbb{S} \otimes -$ (everything p-local).*

Remark 1.24. This is to be contrasted with the fact that $L_{K(h)}$ does not commute with colimits.

The smash product theorem gives us a way to reconstruct $L_h X$ from $L_{h-1}X$ and $L_{K(h)}X$.

Corollary 1.25. *For every p-local spectrum X, there exists a pullback square of the form*

$$\begin{tikzcd}
L_h X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{K(h)}X \arrow[d] \\
L_{h-1}X \arrow[r] & L_{h-1}L_{K(h)}X.
\end{tikzcd}$$

Remark 1.26. This is to be compared with the fracture squares in quasicoherent sheaves obtained from an open-closed decomposition: L_{h-1} is restriction to the open part, while $L_{K(h)}$ is formal completion along the closed part, and $L_{h-1}L_{K(h)}$ is the glueing datum.

Conjecture 1.27 (Weak chromatic splitting conjecture, Hopkins, Hovey)**.** For $X = \mathbb{S}$, or more generally any finite spectrum, the bottom horizontal map

$$L_{h-1}X \to L_{h-1}L_{K(h)}X$$

splits.

Remark 1.28. There is a stronger splitting conjecture, which will be mentioned later. This gives an explicition prediction of what $L_{h-1}L_{K(h)}\mathbb{S}$ should look like, hence allows us to obtain a better description of L_h.

1.5 Morava E-theories

Definition 1.29. Let FG_h be the (1-)category of pairs (Γ_h, k) of a perfect field k of characteristic $p \neq 0$ and a formal group Γ_h over k of height h. The morphisms are given by pairs

$$(\alpha, \beta)\colon (\Gamma_h, k) \to (\Gamma'_h, k')$$

of a ring map $\beta\colon k \to k'$ and an isomorphism $\alpha\colon \Gamma'_h \xrightarrow{\sim} \Gamma_h \otimes_k k'$.

Theorem 1.30 (Goerss–Hopkins–Miller, Lurie)**.** *There exists a functor*

$$E\colon \mathrm{FG}_h \to \mathrm{CAlg}(\mathrm{Sp}_{K(h)})$$

such that the following hold.

1. *The homotopy groups are given by $\pi_* E(\Gamma, k) \cong A(\Gamma, k)[\beta^{\pm}]$, a Laurent series over the Lubin–Tate ring of (Γ, k) in a variable β of degree two.*

2. *Since this ring spectrum is even, it carries a formal group*
$$\mathrm{Spf}(E(\Gamma, k)^0(\mathbb{CP}^\infty))$$
over $E(\Gamma, k)^0 \cong A(\Gamma, k)$, *which can be identified with the universal deformation of* Γ.

Remark 1.31. This should be thought of as a spectral version of the Lubin–Tate ring.

1. By functoriality, $\mathbb{G}_h$ acts on $E_h := E(\Gamma_h, \overline{\mathbb{F}}_p)$ by $\mathbb{E}_\infty$ ring maps.
2. Let $I = (p, u_1, \dots, u_{h-1}) \subset A = A(\Gamma_h, \overline{\mathbb{F}}_p)$ be the augmentation ideal over $W(\overline{\mathbb{F}}_p)$. Then there is a decomposition
$$E_h/I \simeq \bigoplus_{0 \le i \le p^h - 2} \Sigma^{2i} K(h, p).$$
3. The Bousfield localisation functors L_{E_h} and $L_h = L_{K(0) \oplus \cdots \oplus K(h)}$ are equivalent.
4. This construction has further universal properties which we will not discuss.

Theorem 1.32 (Devinatz–Hopkins)**.** *Let us work in* $\mathrm{Sp}_{K(h)}$ *and let* $\widehat{\otimes}$ *denote the localised tensor product on* $\mathrm{Sp}_{K(h)}$, *i.e.* $X \widehat{\otimes} Y = L_{K(h)}(X \otimes Y)$.

1. *We can identify* $E_h \widehat{\otimes} E_h \simeq C_{\mathrm{cts}}(\mathbb{G}_h, E_h)$, *where the right hand side denotes continuous cochains on* $\mathbb{G}_h$, *i.e.*
$$C_{\mathrm{cts}}(\mathbb{G}_h, E_h) = \varinjlim_{\alpha} C(\mathbb{G}_h/H_\alpha, E_h),$$
where H_α *ranges over all finite index subgroups of* $\mathbb{G}_h$.
2. *Let* $E_h^{\widehat{\otimes} \bullet + 1}$ *denote the Amitsur complex of* E_h, *i.e. the cosimplicial object with coface and codegeneracy maps given by multiplications and units. Then the map*
$$L_{K(h)}\mathbb{S} \to \mathrm{Tot}(E_h^{\widehat{\otimes} \bullet + 1})$$
is an equivalence.

Corollary 1.33. *The Bousfield–Kan spectral sequence of the Amitsur complex above therefore takes the form*
$$E_2^{s,t} \cong H^s_{\mathrm{cont}}(\mathbb{G}_h; \pi_t E_h) \implies \pi_{t-s} L_{K(h)}\mathbb{S}.$$

When $h = 1$, this spectral sequence recovers Adams' computations of the image of the J-homomorphism. In general for any (finite) h, there is a horizontal vanishing line on some page. In particular, let us note that $A = \pi_0 E_h$ so that the cohomology groups from the main theorem of the talk appear as $E_2^{*,0}$. Consider the span

$$\begin{tikzcd}
H^*_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \arrow[rr, "\phi"] \arrow[dd] & & H^*_{\mathrm{cont}}(\mathbb{G}_h; \pi_0 E_h) = E_2^{*,0}, \\
 & \widetilde{x}_i \arrow[r, mapsto] & \phi(\widetilde{x}_i) \\
H^*_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & x_i \arrow[u, mapsto] &
\end{tikzcd}$$

where x_i denotes the i-th generator in $H^*_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \Lambda_{\mathbb{Q}_p}(x_1, \dots, x_h)$, and $\widetilde{x}_i$ is any lift of x_i.

Conjecture 1.34 (Strong splitting conjecture)**.** Let p be odd and fix an integer $1 \le i \le h$.

1. $\phi(\widetilde{x}_i) \in E_2^{*,0}$ persists to a nontrivial class
$$e_i \in \pi_{1-2i} L_{K(h)}\mathbb{S}.$$
2. The composite
$$\mathbb{S}^{1-2i} \xrightarrow{e_i} L_{K(h)}\mathbb{S} \to L_{h-1}L_{K(h)}\mathbb{S}$$
factors through $\mathbb{S}^{1-2i} \to L_{h-i}\mathbb{S}^{1-2i}$. The resulting map is denoted
$$\overline{e}_i \colon L_{h-i}\mathbb{S}^{1-2i} \to L_{h-1}L_{K(h)}\mathbb{S}.$$
3. The maps $\overline{e}_i$ induce an equivalence
$$\bigwedge_{i=1}^{n}(L_{h-i}\mathbb{S}^{1-2i}) \xrightarrow{\sim} L_{h-1}L_{K(h)}\mathbb{S}.$$

The left hand side in the final statement is an exterior algebra indexed on the $\mathbb{Z}_p$-module generators of $\Lambda_{\mathbb{Z}_p}(\overline{e}_1, \ldots, \overline{e}_h)$, defined as
$$\bigwedge_{i=1}^{n}(L_{h-i}\mathbb{S}^{1-2i}) = \bigoplus_{0\leq j\leq h} \bigoplus_{1\leq i_1<\cdots<i_j\leq h} \bigotimes_{k=1}^{j} L_{h-i_k}\mathbb{S}^{1-2i_k}.$$

Remark 1.35. The strong chromatic splitting conjecture is known to be true for $h \leq 2$ and $p \geq 3$.

- If $p = 2$, the statement given above is not true, but slight modifications are to be made.
- If the strong conjecture is true, the map
$$L_{h-1}\mathbb{S} \to L_{h-1}L_{K(h)}\mathbb{S}$$
from the chromatic fracture square is the unit of this exterior algebra hence splits (which implies the weak chromatic splitting conjecture).
- Upon rationalisation, the strong chromatic splitting conjecture implies that
$$(\pi_* L_{K(h)}\mathbb{S}) \otimes \mathbb{Q} \cong \Lambda_{\mathbb{Q}_p}(\overline{e}_1, \ldots, \overline{e}_h),$$
where we recall that $\overline{e}_i$ is of degree $1 - 2i$.

1.6 Power operations and splitting

Recall that we want to obtain a splitting in continuous cohomology of the map induced by the inclusion $W(\overline{\mathbb{F}}_p) \to A = \pi_0 E_h$. We denote the latter by E^0 for brevity, the height being implicit.

Definition 1.36. For $m \geq 0$, we define a multiplicative (but not additive!) map
$$P^m \colon E \to E^0(\mathrm{B}\Sigma_m)$$
by
$$\begin{aligned} E^0 =& [\mathbb{S}, E], \\ \to& [(\mathbb{S})^{\otimes m}_{h\Sigma_m}, E^{\otimes m}_{h\Sigma_m}], \\ \to& [\mathbb{S}_{h\Sigma_m}, E] \\ \cong& E^0(\mathrm{B}\Sigma_m), \end{aligned}$$
where we used the multiplication map on E.

Remark 1.37. One easily checks that P^0 is constant at 1, and that P^1 is the identity.

1. P^m is $\mathbb{G}_h$-equivariant for all $m \geq 0$,
2. $E^0(\mathrm{B}\Sigma_m)$ is a free E^0-module of finite rank.
3. $E^0(\mathrm{B}\Sigma_m)$ can be equipped with a unique linear topology compatible with the structure of an E^0-module.

Lemma 1.38. *For $m \geq 0$, the map*

$$P^m \colon E^0 \to E^0(\mathrm{B}\Sigma_m)$$

is continuous with respect to the topology generated by the ideal $I = (p, u_1, \ldots, u_{h-1})$.

Proof. This is a surprisingly difficult lemma. Let $I_{\mathrm{tr}} \subset E^0(\mathrm{B}\Sigma_m)$ denote the transfer ideal, i.e. the ideal generated by the images of the transfers along inclusions of the form $\Sigma_j \times \Sigma_i \subset \Sigma_m$. Then the composite map

$$\overline{P}^m \colon E^0 \to E^0(\mathrm{B}\Sigma_m) \to E^0(\mathrm{B}\Sigma_m)/I_{\mathrm{tr}}$$

is a ring map (i.e. it is furthermore additive) and continuous, with target a finitely generated free E^0-module. We then use Hopkins–Kuhn–Ravenel character theory to embed

$$E^0(\mathrm{B}\Sigma_{p^k}) \to \prod_{i=0}^{k} E^0(\mathrm{B}\Sigma_{p^i})/I_{\mathrm{tr}}. \qquad \square$$

Remark 1.39. As we will see below, the paper uses a $K(h)$-local transfer along the surjective group homomorphism $\Sigma_m \to e$ to obtain a map

$$\mathrm{Tr}^{e}_{\Sigma_m} \colon E^0(\mathrm{B}\Sigma_m) \to E^0.$$

They don't particularly elaborate on this, so Nikolay gave us a sketch of how to construct this: Let $f' \colon G \to H$ be any morphism of finite groups, inducing a map of anima

$$f \colon \mathrm{B}G \to \mathrm{B}H$$

such that the fibre of f can be identified with a coproduct of anima $\mathrm{B}K_i$ for finite groups K_i. Furthermore, f induces an adjunction

$$\begin{tikzcd} \mathrm{LocSys}(\mathrm{B}H; \mathrm{Sp}_{K(h)}) \arrow[r, "f^*" description] & \mathrm{LocSys}(\mathrm{B}G; \mathrm{Sp}_{K(h)}). \arrow[l, bend right, "f_!" description] \arrow[l, bend left, "f_*" description] \end{tikzcd}$$

Due to the higher semiadditivity of $\mathrm{Sp}_{K(h)}$ we see that there is an equivalence $f_! \simeq f_*$, which allows us to produce a transfer map

$$\mathrm{Tr}_f \colon L_{K(h)}\Sigma^\infty_+ \mathrm{B}H \to L_{K(h)}\Sigma^\infty_+ \mathrm{B}G.$$

If we hadn't taken values in $K(h)$-local spectra, this transfer would only exist for injective group homomorphisms.

Proposition 1.40. *There exists a $\mathbb{G}_h$-equivariant continuous map*

$$E^0 \to W(\overline{\mathbb{F}}_p)$$

splitting the inclusion $W(\overline{\mathbb{F}}_p) \to E^0$.

Proof. Set

$$\beta_m \colon E^0 \xrightarrow{P^m} E^0(\mathrm{B}\Sigma_m) \xrightarrow{\mathrm{Tr}^e_{\Sigma_m}} E^0.$$

For every m, these assemble into a map

$$\beta \colon E^0 \to E^0[\![x]\!], \beta(a) = \sum_m \beta_m(a)x^m.$$

Now remark that by the lemma above, β is continuous and $\mathbb{G}_h$-equivariant. Further, since P^0 is constant at the unit (and the transfer map is multiplicative), we see that

$$\beta(a) \in 1 + xE^0[\![x]\!].$$

Since the power operations define an additive map modulo the transfer ideal, we see that $\beta(a+b) = \beta(a)\beta(b)$. This tells us that β defines an additive map

$$\gamma \colon E^0 \xrightarrow{\beta} 1 + xE^0[\![x]\!] \to 1 + x\overline{\mathbb{F}}_p[\![x]\!] \cong W_{\mathrm{big}}(\overline{\mathbb{F}}_p) \to W(\overline{\mathbb{F}}_p).$$

The second map is the quotient by the maximal ideal I in E^0. Note that the last map sits in a further composite

$$1 + x\overline{\mathbb{F}}_p[\![x]\!] \cong W_{\mathrm{big}}(\overline{\mathbb{F}}_p) \to W(\overline{\mathbb{F}}_p) \to \overline{\mathbb{F}}_p$$

with the reduction mod p map from the Witt vectors. This map simply sends a power series to its coefficient at x. Since P^1 acts by the identity, we see that the composite

$$\overline{\mathbb{F}}_p \hookrightarrow E^0 \xrightarrow{\gamma} W(\overline{\mathbb{F}}_p) \to \overline{\mathbb{F}}_p$$

is just the identity. Let f denote the composite $W(\overline{\mathbb{F}}_p) \hookrightarrow E^0 \xrightarrow{\beta} W(\overline{\mathbb{F}}_p)$. Then our observation that $\gamma(x)$ reduces to x modulo p tells us that f is a homomorphism of p-adically complete abelian groups which reduces to the identity modulo p hence must be an isomorphism. Then define $\alpha := f^{-1} \circ \gamma$ as a map $E^0 \to W(\overline{\mathbb{F}}_p)$. Per construction this is a $\mathbb{G}_h$-equivariant, additive, continuous section of the inclusion. □

2 Adic spaces (Ningchuan Zhang, 17 June)

The main references for this talk are section 1 of J. Weinstein's lectures at the 2017 AWS ([Wei17]) and sections 2 and 3 of Scholze–Weinstein ([SW13]).

2.1 Motivation

Last time, we proved that we have a split injection of the form

$$H^*_{\mathrm{cont}}(\mathbb{G}_h; W) \longhookrightarrow H^*_{\mathrm{cont}}(\mathbb{G}_h; A),$$

where $W = W(\overline{\mathbb{F}}_p)$ and $A = \pi_0 E(\Gamma_h, \overline{\mathbb{F}}_p)$ is the Lubin–Tate ring. The goal of introducing adic spaces is to be able to describe why this is a rational isomorphism.

Remark 2.1. Recall that a stronger result is conjectured. Namely the Vanishing conjecture (Conjecture 1.11) states that the split injection above is an isomorphism before rationalisation.

Note that the right hand side of this split injection can be described conceptually as the sheaf cohomology of the formal stack

$$\mathrm{Spf}(A) /\!\!/ \mathbb{G}_h.$$

However, the rational continuous cohomology of $\mathbb{G}_h$ does not admit such a description in terms of sheaf cohomology over (a quotient of) a formal scheme. Indeed formal schemes are set up such that the global sections of their structure sheaf is always complete for the topology on the underlying ring, but it is clear that $\mathbb{Q}_p$ is not p-complete. Adic spaces therefore give us a way of simultaneously incorporating the generic fibre as geometric information. To describe the right hand side, we will explicitly need to understand the geometry of the adic space $\mathrm{LT}^{\mathrm{ad}}_{W\otimes_{\mathbb{Z}_p}\mathbb{Q}_p}$ obtained as the generic fibre of the Lubin–Tate space.

Remark 2.2. Adic spaces form a general framework that should encompass classical schemes, formal schemes, and rigid analytic spaces. In particular, it should respect the inclusion of schemes into formal schemes an the construction of a rigid analytic space from a formal scheme.

Let us begin with a philosophical question: what is a space? The answer we follow in the construction of adic spaces is that it is a topological space (i.e. set of points with a topology) with a structure sheaf of rings. In some cases, we will not actually have a topological space, but rather a site.

Example 2.3.

- Smooth manifolds are such an example, which are locally isomorphic to an open subset of $U \subset \mathbb{R}^n$, with structure sheaf given by $U \mapsto C^\infty(U)$.
- Schemes are an example as well, being locally isomorphic to $\mathrm{Spec}(A)$ for a ring A and with structure sheaf $\mathcal{O}_A$ as usual.
- An adic space will be a topologically ringed space with valuations on the stalks, which is locally of the form $\mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+).

2.2 Formal schemes

Remark 2.4. All ideals of definition in a topological ring will be assumed to be finitely generated throughout.

Definition 2.5. A (linearly) topological ring A is called *adic* if there exists an open ideal $I \subset A$, called an *ideal of definition*, such that the open subsets $\{I^n\}_{n\geq 0}$ form a basis of open neighbourhoods of the point $0 \in A$. Further, A is assumed to be complete and separated with respect to this topology.

Example 2.6.

- We can let A be a discrete ring and $I = (0)$.
- Let $A = \mathbb{Z}_p$ be the p-adic integers with the usual p-adic topology, then we can choose $I = (p^k)$ for any $k \geq 1$.

Remark 2.7. Two ideals of definition I, J generate the same topology if and only if their radicals agree, i.e. $\sqrt{I} = \sqrt{J}$.

Example 2.8. Let $A = \mathbb{Z}_p[\![T]\!]$. Then A can be viewed as an adic ring in several distinct ways, namely

$$I = (p), \qquad I = (T), \qquad I = (p, T).$$

In either case, A will be complete for the I-adic topology, but it is clear that their radicals are different.

Definition 2.9. Let A be an adic ring. Then define $\mathrm{Spf}(A)$ as the set of open prime ideals of A with topology generated by the basic opens

$$D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$$

as f ranges over elements of A and structure sheaf defined by

$$\Gamma(D(f); \mathcal{O}_{\mathrm{Spf}(A)}) = (A[f^{-1}])^{\wedge}_{I}.$$

Remark 2.10. Note that in particular the global sections of a formal scheme are always I-complete, which precludes obtaining a non p-complete $\mathbb{Z}_p$-module like $\mathbb{Q}_p$ as global sections of some p-adic formal scheme as mentioned in the motivation section.

Remark 2.11. The open condition on the prime ideals changes the underlying topological space. In the case of $\mathbb{Z}_p$ we have

$$\mathrm{Spec}(\mathbb{Z}_p) = \{(0), (p)\}, \qquad \mathrm{Spf}(\mathbb{Z}_p) = \{(p)\}.$$

In fact, $\mathrm{Spf}(A)$ is homeomorphic to $\mathrm{Spec}(A/I)$ but the structure sheaf accounts for the additional "fuzz".

Definition 2.12. A formal scheme is a locally ringed space which is locally isomorphic to $\mathrm{Spf}(A)$ for an adic ring A.

2.3 Rigid spaces

Let K be a non-archimedean field, so that it is in particular complete with respect to some valuation denoted $|\cdot|$. Examples include $\mathbb{Q}_p$, where $|p| = p^{-1}$.

Definition 2.13. The *Tate algebra* $K\langle T_1, \dots, T_n\rangle$ is the completion of the polynomial ring $K[T_1, \dots, T_n]$ with respect to the *Gauß norm*. The latter is defined by

$$\left\| \sum_I a_I T^I \right\| = \sup_I |a_I|.$$

Elements of the Tate algebra are to be thought of as formal power series $f = \sum_I a_I T^I$ such that the sequence $|a_I|$ converges to zero.

Definition 2.14. A *K-affinoid algebra* is a quotient of the Tate algebra over K by some closed ideal[1]. For a K-affinoid algebra A, we define the associated *affinoid space* by the following.

- As a set, it is given by $\mathrm{mSpec}(A)$, the set of maximal ideals in A.
- The topology is generated by the rational opens. For $x \in \mathrm{mSpec}(A)$ a maximal idea, A/x will be a finite extension of K, so that the valuation on K extends uniquely to a valuation on this residue field. For an element $f \in A$, write $|f(x)|$ for the norm of f in A/x. For $f_1, \dots, f_n, g \in A$ we then define the associated rational open by

$$D\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in \mathrm{mSpec}(A) \mid \forall i = 1, \dots, n, |f_i(x)| \le |g(x)|\}.$$

---

[1] Actually, all ideals of the Tate algebra are closed, ([Wei17, p. 3]).

- The structure sheaf is defined on rational opens by
$$\Gamma\left(D\left(\frac{f_1,\dots,f_n}{g}\right);\mathcal{O}\right)=A\langle T_1,\dots,T_n\rangle/(f_i-gT_i\mid i=1,\dots n).$$

Remark 2.15. This need not always form a topological space, but in general the rational opens form a basis for a site.

Definition 2.16. A *rigid analytic space* over K is a locally topologically ringed space that is locally isomorphic to the affinoid space associated to a K-affinoid algebra.

Example 2.17.

- The affinoid space associated to the $\mathbb{Q}_p$-affinoid algebra $\mathbb{Q}_p\langle T\rangle$ is called the rigid closed disc. Indeed, if $K/\mathbb{Q}_p$ is a finite extension, we see that its functor of points satisfies
$$\mathrm{mSpec}(\mathbb{Q}_p\langle T\rangle)(K)=\{x\in K\mid\{x^n\}_n \text{ is bounded}\}.$$
The latter is equivalently the set $\{x\in K\mid |x|\le 1\}$ whence the name.
- Given a finite type formal scheme, we can construct an analytic space by taking its generic fibre.

Let us now mention an application of the theory of rigid analytic spaces in chromatic homotopy theory, given by the Gross–Hopkins period map.

Theorem 2.18 (Gross–Hopkins, [HG94])**.** *There is a $\mathbb{G}_h$-equivariant map*
$$\pi_{GH}\colon \mathrm{LT}_K^{\mathrm{rig}}\longrightarrow\mathbb{P}_K^{h-1}$$
of rigid analytic spaces which is étale surjective and such that the pullback $\pi_{GH}^*\mathcal{O}(1)$ is the Lie algebra of the universal deformation Γ_h over $\mathrm{LT}_K^{\mathrm{rig}}$.

This is used to study Brown–Comenetz duality in $K(n)$-local spectra by Strickland in [Str00].

2.4 Huber rings

Definition 2.19. A *Huber ring* is a topological ring A such that one can find an open subring $A_0\subset A$ carrying the I-adic topology for some finitely generated ideal $I\subset A_0$. A Huber ring A is said to be *Tate* if A contains a topologically nilpotent unit, called the *pseudo-uniformiser.*

Example 2.20.

1. We can let $A=A_0$ be a discrete ring and $I=(0)$.
2. We can let $A=A_0$ be an I-adically complete ring for $I\subset A$ a finitely generated ideal of definition.
3. Let $A=K$ be a non-archimedean field, then we can set $A_0=\mathcal{O}_K=\{x\in K\mid |x|\le 1\}$. This admits a pseudo-uniformiser ϖ given by any element of $\mathcal{O}_K$ such that $0<|\varpi|<1$.
4. Let $A=K\langle T_1,\dots,T_n\rangle$ be the Tate algebra over K, and set $A_0=\mathcal{O}_K\langle T_1,\dots,T_n\rangle$. This has a pseudo-uniformiser given by the same ϖ as above.
5. Let K be a non-archimedean perfect field of characteristic p, and set $A=A_0=W(\mathcal{O}_K)$ where now $I=(p,[\varpi])$ for $[\varpi]$ the multiplicative lift of $\varpi\in\mathcal{O}_K$.
6. For a non-example, note that $A=\mathbb{Q}_p[\![T]\!]$ is not a Huber ring. One would like to set $A_0=\mathbb{Z}_p[\![T]\!]$, but this subring is not open. Indeed, the sequence $\{p^{-1}T^n\}$ is not contained in A_0 but it converge to $0\in A_0$.

Remark 2.21. In the examples above, the first two are not Tate, while the next three are.

2.5 The underlying topological space

For a Huber ring A, we want to set $\mathrm{Spa}(A)$ to be a subset of the set of continuous valuations on A. We will see below that the actual definition is more involved, but let us first note why this is a sensible definition. Recall that if K is a field with a subring A, we defined the Zariski–Riemann space $\mathrm{Zar}(K, A)$ of the pair (K, A) to be the space of continuous valuations on K such that for all $a \in A$, $|a| \leq 1$. This defined a quasicompact ringed space, and it is actually a scheme in nice cases. For example, if K/k is a field extension of transcendence degree one, then

$$\mathrm{Zar}(K, k) \cong \text{smooth projective curve with function field } K \text{ over } k.$$

The easiest example is $K = k(x)$, which corresponds to $\mathbb{P}^1_k$.

Remark 2.22. Note that we do not restrict the target of our valuations, they can land in any ordered abelian group. Further, note that if Γ is an ordered abelian group, then $\Gamma \cup \{0\}$ is an ordered monoid with minimum 0. Examples of such ordered abelian groups include $\mathbb{R}^{\times n}_{>0}$ for $n \geq 1$ with multiplication and lexicographical order.

Definition 2.23. A *continuous valuation* on a topological ring A is a continuous map

$$|\cdot|\colon A \longrightarrow \Gamma \cup \{0\}$$

such that

- $|ab| = |a||b|$,
- $|a + b| \leq \max(|a|, |b|)$,
- $|1| = 1$,
- $|0| = 0$, and
- for all $\gamma \in \Gamma$, the set $\{a \in A \mid |a| < \gamma\} \subset A$ is open.

As seems to be standard, we will usually denote valuations by $x\colon A \to \Gamma \cup \{0\}$, and write $|f(x)| \coloneqq x(f)$ for $f \in A$.

Definition 2.24. Given a topological ring A, we define the set $\mathrm{Cont}(A)$ of equivalence classes of continuous valuations on A: the *value group* of a valuation $x\colon A \to \Gamma \cup \{0\}$ is the subgroup of Γ generated by the image $x(A)$, and often we implicitly replace $\Gamma \cup \{0\}$ with the value group (though even in this case $|-|$ need not be surjective unless A is a field). The equivalence relation between valuations identifies x and x' if there is a commutative diagram

$$\begin{tikzcd}
 & \Gamma \cup \{0\} \arrow[d, "\cong"] \\
A \arrow[ur, "x"] \arrow[r, "x'"'] & \Gamma' \cup \{0\}
\end{tikzcd}$$

where Γ, Γ' are the value groups and the vertical map is an order preserving isomorphism.

Let us now topologise $\mathrm{Cont}(A)$ with a basis given by the rational opens

$$D\left(\frac{f_1, \ldots, f_n}{g}\right) = \{x \in \mathrm{Cont}(A) \mid |f_i(x)| \leq |g(x)| \neq 0 \text{ for } i = 1, \ldots, n\}.$$

Remark 2.25. Note that in the definition above, setting $f_i = 1$ recovers the Zariski opens, while allowing $g = 0$ recovers the basis of opens in the affinoid rigid analytic space associated to A.

Definition 2.26. A *Huber pair* is a pair of a Huber ring A and a subring A^+ of integral elements such that every element of A^+ is power bounded, and $A^+ \subset A$ is open and integrally closed. Define the space

$$\operatorname{Spa}(A, A^+)$$

to be the subspace of $\operatorname{Cont}(A)$ on valuations x such that for all $f \in A^+$, $|f(x)| \leq 1$.

Remark 2.27. The subring of power-bounded elements of a Huber ring is denoted A°. In fact, we see that A° is the union of all possible choices of $A_0 \in A$.

Example 2.28. If C is a non-archimedean algebraically closed field, such as $\mathbb{C}_p$, then $X = \operatorname{Spa}(C\langle T\rangle, C^\circ\langle T\rangle)$ has five types of points.

1. For any $\alpha \in C$, such that $|\alpha| \leq 1$, so that in particular $\alpha \in C^\circ$, we obtain a point
$$f \longmapsto |f(\alpha)|.$$
2. Let α be as above, and let $D = D(\alpha, r)$ be the disc centred at α with radius $0 < r < 1$ in the image of the valuation map on C. Then we obtain a point by
$$f \longmapsto \sup\{|f(\beta)| \mid \beta \in D\}.$$
3. The construction above also works when $0 < r < 1$ is not in the image of the valuation map on C.
4. If C is not spherically complete (e.g. when $C = \mathbb{C}_p$), i.e. there exists a decreasing sequence of discs with radius < 1 of the form
$$D_1 \supset D_1 \supset D_3 \supset \cdots$$
such that $\bigcap_i D_i \neq \varnothing$, then we obtain a point by
$$f \longmapsto \inf_{i \geq 1} \sup_{\beta_i \in D_i} |f(\beta_i)|.$$
5. Let α be as above, and now $0 < r \leq 1$. Pick a sign $\pm$ (excluding $+$ if $r = 1$) and let $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, be the ordered abelian group generated by $\mathbb{R}_{>0}$ and an element γ which is infinitesimally less than or greater than r (depending on the sign we chose). Then we obtain a point by
$$f = \sum_{n=0}^{\infty} a_n (T - \alpha)^n \longmapsto \sup_n |a_n| \gamma^n.$$

Remark 2.29. Points of type 2 or 3 are called Gaußpoints, while the points of type 5 are said to be of rank two.

2.6 The structure sheaf

Let $U \subset X \coloneqq \operatorname{Spa}(A, A^+)$ be a rational open subset, we want to specify the value $\Gamma(U; \mathcal{O}_X)$. This is done perhaps rather indirectly using the following theorem of Huber.

Theorem 2.30 (Huber). *Let U and (A, A^+) be as above, then there is a complete Huber pair with a map*

$$(A, A^+) \longrightarrow (\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$$

such that the induced map on adic spectra factors through $U \subset X$ and the induces map

$$\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \longrightarrow U$$

is a homeomorphism, and terminal among all such factorisations. In particular, U is quasi-compact.

Remark 2.31. The hard part is not defining $\mathcal{O}_X(U)$, since this is defined in the same way as before, i.e. for $U = D(\frac{f}{g})$, we see that $\mathcal{O}_X(U) = A\langle T\rangle/(fT - g)$. However, $\mathcal{O}_X^+(U)$ is the integral closure of $A^+[\frac{f}{g}]$ in this ring, which is less evident. We will see below that $\mathcal{O}_X^+(U)$ can be recovered from $\mathcal{O}_X(U)$.

Definition 2.32. Let $X = \mathrm{Spa}(A, A^+)$ be the adic spectrum of a Huber pair as above. Define a presheaf on X by sending an open $W \subset X$ to

$$\mathcal{O}_X = \varprojlim_{U \subset W} \mathcal{O}_X(U),$$

where the limit ranges over rational opens U contained in W. We say that a Huber pair $(A, A+)$ is *sheafy* if this presheaf is actually a sheaf on X.

Proposition 2.33. *We can identify*

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \mid \forall x \in U, |f(x)| \leq 1\}.$$

In particular, this is a sheaf if $\mathcal{O}_X$ is. Furthermore, if (A, A^+) is a complete Huber pair, then on global sections we obtain

$$\mathcal{O}_X(X) = A, \qquad \mathcal{O}_X^+(X) = A^+.$$

Definition 2.34. An *adic space* is a triple $(X, \mathcal{O}_X, |\cdots|_{x\in X})$ of a topologically ringed space equipped with valuations on its stalks, such that it is locally of the form $\mathrm{Spa}(A, A^+)$ for a sheafy Huber pair.

Remark 2.35. We see that a Huber pair (A, A^+) is sheafy if

- $A = A^+$ is discrete,
- $A = A^+$ is finitely generated over a Noetherian ring, or
- A is Tate and such that the Tate algebras $A\langle T_1, \dots, T_n\rangle$ are Noetherian for all $n \geq 0$.

We conclude that an affine scheme $\mathrm{Spec}(A)$ can be viewed as an adic space $\mathrm{Spa}(A, A)$, a formal scheme $\mathrm{Spf}(A)$ can be viewed as an adic space $\mathrm{Spec}(A, A)$, and a K-affinoid rigid analytic space $\mathrm{mSpec}(A)$ can be viewed as an adic space $\mathrm{Spa}(A, A^\circ)$.

3 Examples of adic spaces (Itamar Mor, 24 June)

3.1 Complements

Recall that a Huber ring A is a topological ring admitting an open I-adic subring A_0 for some $I \trianglelefteq A_0$. A subset $T \subseteq A$ is call *bounded* if for any open subset $U \subseteq A$, there is an open subset $V \subseteq A$ such that $TV \subseteq U$. This definition makes sense for any topological ring, and in the Huber case happens if and only if for any ideal I of definition of A and $n \geq 0$, $I^m T \subseteq I^n$ for $m \gg 0$.

Definition 3.1. We define the subsets of *power bounded* and *topologically nilpotent* elements of a topological ring A as:

$$A^\circ := \{f \in A \mid f^{\mathbb{N}} \text{ is bounded}\},$$
$$A^{\circ\circ} := \{f \in A \mid f^n \to 0\}.$$

Also recall a Huber pair (A, A^+) is a Huber ring A together with an integrally closed and open subring A^+.

Observation 3.2. Here are some facts about Huber rings and pairs:

1. Any ideal of definition I is contained in $A^{\circ\circ}$ since $I^n \to 0$.
2. Any ring of definition A_0 is bounded and therefore contained in A°.
3. The set A° itself does not need to be bounded. For example, take $A = \mathbb{Q}_p[T]/T^2$. Then $A^\circ = \mathbb{Z}_p \oplus \mathbb{Q}_p\{T\}$ is not bounded.
4. In fact, $A^\circ = \bigcup A_0$ is the union of all rings of definition in A. Essentially, for any power bounded element $f \in A$, there is a ring of definition A_0 containing f (in fact, this is a filtered colimit).
5. For any Huber pair (A, A^+), we have $A^{\circ\circ} \subseteq A^+$. This is because for any topologically nilpotent element f, we have $f^N \in A^+$ when $N \gg 0$ since $f^n \to 0$ and A^+ is open. The element f is then contained in A^+, since A^+ is integrally closed.

Definition 3.3. A Huber ring A is called *Tate* if there is an element $\varpi \in A^{\circ\circ} \cap A^\times$. Such ϖ is called a *pseudo-uniformizer*. If we fix a ring of definition, we may assume without loss of generality that $\varpi \in A_0$ (replace ϖ by some power) and $I = (\varpi)$, in which case and $A = A^+[\varpi^{-1}]$.

3.2 Completeness and sheafiness

Next, we will talk a bit about sheafiness. To this end, we first explain the role of completeness in Huber rings.

We've defined the space $\operatorname{Spa}(A, A^+)$, and want to equip it with a structure sheaf $\mathcal{O}$. In (usual) algebraic geometry, we do so by localising, and one key point for doing so is Nullstellensatz: for a discrete ring A there are one-to-one correspondences

$$\{\text{Radical ideals of } A\} \longleftrightarrow \{\text{Closed subsets of } \operatorname{Spec} A\} \longleftrightarrow \{\text{Open subsets of } \operatorname{Spec} A\}$$

the right-hand given by taking complements and the left by $I \mapsto V(I)$ and $V \mapsto I(V)$. In turn, this depends on the fact that $\operatorname{Spec} A = \emptyset$ if and only if $A = 0$. For example, one consequence is that we can recover the units from $\operatorname{Spec} A$:

Corollary 3.4. *If $f \in A$ has $\varphi(f) \neq 0$ for any map $\varphi\colon A \to k$ to a field, then $f \in A^\times$.*

To define $\mathcal{O}$, we want a similar picture with valuations replacing prime ideal. The starting point is:

Theorem 3.5 ([Hub93], Proposition 3.6)**.** *Given a Huber pair (A, A^+), recall* $\operatorname{Spa}(A, A^+)$ *is the set of equivalence classes of continuous valuations on A such that $|f(x)| \leq 1$ for any $f \in A^+$. We have*

- $\operatorname{Spa}(A, A^+) = \emptyset \iff A/\overline{\{0\}} = 0$. *In particular, $A = 0$ if it is Hausdorff.*

- $A^+ = \{f \in A \mid |f(x)| \leq 1, \ \forall x \in \mathrm{Spa}(A, A^+)\}$.

See also the notes by Morel.

Corollary 3.6 ([SW20], Proposition 2.3.10)**.** *Let (A, A^+) be a Huber pair.*

1. *An element $f \in A$ lives in A^+ if and only if $|f(x)| \leq 1$ for any $x \in \mathrm{Spa}(A, A^+)$*
2. *If (A, A^+) is* complete *and* Hausdorff*, then*

$$|f(x)| \neq 0, \ \forall x \quad \Longrightarrow \quad f \in A^\times.$$

Proof. Suppose $A \neq 0$ and $f \notin A^\times$. Then f must be contained in some (proper) maximal ideal $\mathfrak{m}$. We claim $\mathfrak{m}$ is closed. Then $A/m \neq 0$ is Hausdorff and nonzero. Picking any valuation $x \in \mathrm{Spa}(A/\mathfrak{m}, A^+/\mathfrak{m})$ and extending to A, we have $|f(x)| = 0$.

To see $\mathfrak{m}$ is closed, we notice that $A^{\circ\circ} \subseteq A$ is open since it contains the ideal of definition I. Then $A^\times$ is also open as it contains an open subset $1 + A^{\circ\circ}$ (since A is complete). We then have $\mathfrak{m}$ is contained in the closed subset $A \backslash A^\times$. Therefore its closure $\overline{\mathfrak{m}}$ must be a proper subset of A. One can further check that $\overline{\mathfrak{m}}$ is also an ideal. Hence, $\overline{\mathfrak{m}} = \mathfrak{m}$ since it is a maximal ideal. ☐

Remark 3.7. 1. Henceforth, 'complete' will mean 'Hausdorff and complete'.

2. We can weaken the completeness assumption, for example to allow Henselian A—see the notes by Bhatt. However, we will see later that we have to make *some* restriction.

The following result tells us how to compute the topological completion from the adic completion:

Lemma 3.8 ([Hub93], Lemma 1.6)**.** *Let A_0 be a ring of definition for A. Then $\widehat{A} \cong \widehat{A}_0 \otimes_{A_0} A$.*

Warning 3.9. In general, Huber rings/pairs do not have pushouts. The situation is better for pushouts along *adic* maps, that is, maps of Huber rings $\varphi \colon A \to B$ for which there is an ideal of definition I such that $\varphi(I)$ generates the topology on B. Then Huber proves under finiteness assumptions that the pushout exists [Hub96, Proposition 1.2.2].

Let (A, A^+) be a Huber pair. Recall rational open subsets of $\mathrm{Spa}(A, A^+)$ are defined to be:

$$U = U\left(\frac{f_1, \cdots, f_n}{g}\right) = \{x \in \mathrm{Spa}(A, A^+) \mid |f(x)| \leq |g(x)| \neq 0\}.$$

We want to define the sections $\mathcal{O}(U)$ functorially in U. One idea is that we should have $g \in \mathcal{O}(U)^\times$ and $\frac{f_i}{g} \in \mathcal{O}^+(U)$. A natural guess is then to define

$$(B, B^+) = \left(A\left[1/g\right], \overline{A^+[f_1/g, \cdots, f_n/g]}\right), \tag{3.10}$$

where $\overline{A^+[f_1/g, \cdots, f_n/g]}$ is the integral closure of $A^+[f_1/g, \cdots, f_n/g]$ in $A[1/g]$.

Warning 3.11. This is not obviously independent of f_i and g. In fact, we will see this definition is *not* independent of f_i and g later in Example 3.25.

Given a map of Huber pairs $\varphi \colon (A, A^+) \to (C, C^+)$, denote by $\varphi^\# \colon \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(A, A^+)$ the induced map on adic spaces. When $\mathrm{Im}(\varphi^\#) \subseteq U$, we want a factorization:

$$\begin{tikzcd}
(A, A^+) \arrow[rr, "\varphi"] \arrow[dr] & & (C, C^+) \\
& (\mathcal{O}(U), \mathcal{O}^+(U)) \arrow[ur, dashed, "\exists!\ \overline{\varphi}"'] &
\end{tikzcd}$$

Theorem 3.12. *If (C, C^+) is a complete Huber pair, there exists a unique factorization $\overline{\varphi}$.*

Proof. To get a map $\overline{\varphi}\colon B \to C$ we need to show that $\varphi(g) \in C^\times$; what we know is that for any $x \in \mathrm{Spa}(C, C^+)$,

$$|\varphi(g)(x)| = |g(\varphi^\# x)| \neq 0,$$

since $\varphi^\#$ lands in U. Since C is complete, we deduce that $\varphi(g) \in C^\times$. To check that $\overline{\varphi}$ restricts to $B^+ \to C^+$ we need to check that $|f_i/g(x)| \leq 1$ for each i and $x \in \mathrm{Spa}(C, C^+)$, which again follows from Corollary 3.6 ☐

Lemma/Definition 3.13 ([Mor19] Lemma III.4.2.3)**.** *Let (B, B^+) be a Huber pair. The completion of B^+ defines a ring of integral elements of $\widehat{B}$, and the pair $(\widehat{B}, \widehat{B}^+)$ is the* completion *of (B, B^+).*

Corollary 3.14. *The pair $(\widehat{B}, \widehat{B}^+)$ has the universal property that it is initial among* complete *Huber pairs (C, C^+) such that $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(A, A^+)$ factors through $U = U\left(\frac{f_1, \ldots, f_n}{g}\right)$.*

Definition 3.15. Set $(\mathcal{O}(U), \mathcal{O}^+(U)) := (\widehat{B}, \widehat{B}^+)$. This depends only on the rational subset U, and so defines a presheaf on $\mathrm{Spa}(A, A^+)$. Recall that (A, A^+) is called *sheafy* if $\mathcal{O}$ happens to be a sheaf.

As a sanity check, we have:

Theorem 3.16 ([Hub93], Proposition 3.9)**.** *Completion induces a homeomorphism* $\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \cong \mathrm{Spa}(A, A^+)$.

Therefore, we may for most purposes replace an arbitrary Huber pair by its completion. The payoff for introducing completion in the definition of $\mathcal{O}$ is that in full generality it ruins any chance of $\mathcal{O}$ remaining a sheaf. The following omnibus theorem (see [SW20, Theorem 3.1.8 and Theorem 5.2.5] or [Mor19, Theorem IV.1.1.5]) nevertheless gives sheafiness in all cases we care about:

Theorem/Definition 3.17 ([Hub94], [BV18], [KL15],...)**.** *Let (A, A^+) be a complete Huber pair. The assignment $U \mapsto \mathcal{O}(U)$ is a sheaf in the following cases:*

1. *A is discrete.*
2. *A is finitely generated over a Noetherian ring of definition A_0.*
3. *A is Tate and* strongly Noetherian*: that is, the Tate algebras $A\langle X_1, \cdots, X_n\rangle$ are Noetherian for all $n \geq 0$* [2]*.*
4. *A is* stably uniform*: that is, A° is bounded (such A is called uniform) and the same is true for $\mathcal{O}(U)^\circ$ for every rational subset U.*

Example 3.18. Theorem 3.17 covers the following cases of interest:

- $\mathbb{Q}_p$ satisfies item (2).
- $\mathbb{C}_p$ satisfies item (3).
- Perfectoids satisfy item (4).

---

[2] Caution: Unlike polynomial algebras, the Hilbert Basis Theorem fails for Tate algebras $A\langle - \rangle$.

Remark 3.19. If a Huber pair (A, A^+) is sheafy, then for any $x \in \mathrm{Spa}(A, A^+)$, we get a valuation $|\cdot|_x$ on the stalk

$$\mathcal{O}_{X,x} \coloneqq \operatorname*{colim}_{U \ni x} \mathcal{O}(U)$$

Then for $f \in \mathcal{O}(U)$, we have

$$f \in \mathcal{O}^+(U) \quad \iff \quad |f(x)|_x \leq 1 \text{ at } \mathcal{O}_{X,x} \text{ for any } x.$$

We are now ready to define the category of adic spaces.

Definition 3.20. Consider the category $\mathcal{V}$ with

- objects are triples $(X, \mathcal{O}_X, (|\cdot|_x)_{x \in X})$.
- morphisms are maps of topologically ringed spaces that are compatible with the valuations on stalks.

Exercise 3.21. Check that any such ringed space is locally ringed, and that any such morphism is a map of locally ringed spaces.

Definition 3.22. An object $X \in \mathcal{V}$ is called an *adic space* if X is locally of the form $\mathrm{Spa}(A, A^+)$ for some *sheafy* Huber pair (A, A^+). It is called *pre-adic* if (A, A^+) is not necessarily sheafy. Denote by CAff the category of sheafy complete Huber pairs, where C stands for “complete”.

Theorem 3.23 ([Hub94], Proposition 2.1)**.** *The adic space construction gives a fully faithful embedding* $\mathrm{Spa}\colon \mathrm{CAff}^{\mathrm{op}} \hookrightarrow \mathcal{V}$*. This gives rise the functor of points for adic spaces:*

$$\begin{aligned} \mathrm{AdicSp} &\hookrightarrow \mathrm{Sh}(\mathrm{CAff}^{\mathrm{op}}) \\ X &\mapsto \mathrm{Hom}(\mathrm{Spa}(-), X), \end{aligned}$$

where $\mathrm{CAff}^{\mathrm{op}}$ *has a “Zariski” topology.*

This perspective will be very useful in computing with adic spaces. Moreover, any pre-adic space also has a functor of points, though now the assignment is not fully faithful; in some cases, we will nevertheless use pre-adic spaces as auxiliary spaces when identifying adic spaces.

3.3 Examples

Now we switch gears and work out some examples.

3.3.1 The terminal object

The first example of adic spaces is the terminal object $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$. As a set, this adic space contains the following points:

- For each prime p, there is a closed point $x_p\colon \mathbb{Z} \to \mathbb{F}_p \to \{0, 1\}$, where $|n|_{x_p} = 0$ iff $p \mid n$.
- For each prime p, there is another point $\eta_p\colon \mathbb{Z} \to \mathbb{Q}_p \to p^{\mathbb{Z}} \cup \{0\}$, where $|n|_{\eta_p} = p^{-v_p(n)}$. The closure of η_p is $\{\eta_p, x_p\}$.
- A generic point $\eta\colon \mathbb{Z} \to \mathbb{Q} \to \{0, 1\}$, where $|n|_\eta = 0$ iff $n = 0$.

Below is a picture of $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$, where each squiggly arrow denotes a specialization and the blue paths encircle closed subsets of $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$. Note that as a topological space, $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Spec}\, \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.

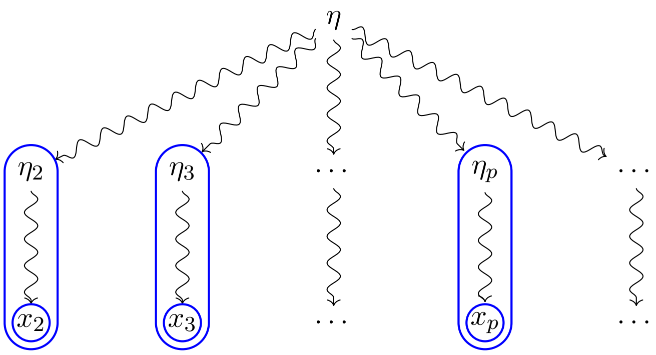

Figure 1: $\mathrm{Spa}(\mathbb{Z},\mathbb{Z})$

3.3.2 $\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)$

Consider a complete local non-archimedean local field K with ring of integers $\mathcal{O}_K$ and a uniformizer ϖ. Then we have:

$$\begin{tikzcd}
\mathrm{Spa}(K,\mathcal{O}_K) \arrow[d, hook] & = & \{\eta_\varpi\} \arrow[d, mapsto] & \cong & \mathrm{Spec}\,K \arrow[d, hook] \\
\mathrm{Spa}(\mathcal{O}_K,\mathcal{O}_K) & = & \{\eta_\varpi \rightsquigarrow x_\varpi\} & \cong & \mathrm{Spec}\,\mathcal{O}_K,
\end{tikzcd}$$

where $|\varpi(\eta_\varpi)| = \varpi^{-1}$ and $|\varpi(x_\varpi)| = 0$ and the $\cong$'s on the right hand side are isomoprhisms of locally ringed spaces. In this sense, $\mathrm{Spa}(K,\mathcal{O}_K) \hookrightarrow \mathrm{Spa}(\mathcal{O}_K,\mathcal{O}_K)$ is the inclusion of the generic point. Restricting along this map (e.g. at the level of functors of points), we get the generic fiber.

Remark 3.24.
- A map of Huber pairs $\varphi\colon (\mathcal{O}_K,\mathcal{O}_K) \to (A,A^+)$ is the condition that $\varphi(\varpi) \in A$ is topologically nilpotent, and this factors through $(K,\mathcal{O}_K)$ precisely when $\varphi(\varpi)$ is also invertible. In this case $\varphi(\varpi) \in A^{\circ\circ} \cap A^\times$, and hence A is Tate.
- The above examples indicate a general phenomenon: for any Huber pair (A,A^+) we have maps

  $$\begin{aligned} \mathrm{Spa}(A,A^+) &\longrightarrow \mathrm{Spec}\,A, & x &\longmapsto \ker(x), \\ \mathrm{Spec}\,A &\longrightarrow \mathrm{Spa}(A^\delta,A^\delta), & \mathfrak{p} &\longmapsto (A \to A_\mathfrak{p}/\mathfrak{p} \to \{0,1\}). \end{aligned}$$

  This exhibits $\mathrm{Spec}\,A$ as a retract of $\mathrm{Spa}(A^\delta,A^\delta)$ (as topological spaces).

3.3.3 Closed unit disc

Consider $\mathbb{D} := \mathrm{Spa}(\mathbb{Z}[T],\mathbb{Z}[T])$. Define

$$\begin{aligned} \mathbb{D}_K &:= \mathbb{D} \times_{\mathrm{Spa}(\mathbb{Z},\mathbb{Z})} \mathrm{Spa}(K,\mathcal{O}_K) \\ &\cong \mathrm{Spa}(K\langle T\rangle,\mathcal{O}_K\langle T\rangle). \end{aligned}$$

This is the *rigid analytic closed disc* [3], and is an affinoid space if either K is strictly noetherian or $\mathcal{O}_K$ is noetherian. To see the isomorphism claimed above we used the functor of points: for

---

[3] For example, its coordinate ring is analogous to the ring of *overconvergent* holomorphic functions on the closed unit disc in $\mathbb{C}$.

any complete Huber pair (A, A^+) we have $\mathbb{D}(A, A^+) = A^+$, and if $(K, \mathcal{O}_K) \to (A, A^+)$ then in particular A is Tate and hence

$$\mathbb{D}_K(A, A^+) = A^+ = \hom((K\langle T\rangle, \mathcal{O}_K\langle T\rangle), (A, A^+)).$$

Here we used completeness of A in the factorization:

$$\begin{tikzcd}
K[T] \arrow[rr] \arrow[dr] & & A \\
& K\langle T\rangle \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

Example 3.25 (Counterexample)**.** Here is an example where (B, B^+) in (3.10) depends on $f_1, \cdots, f_n, g$. Take $(A, A^+) = (\mathbb{Q}_p[T], \mathbb{Z}_p[T])$ with the p-adic topology on $\mathbb{Z}_p[T]$ and consider $f = 1 + pT \notin A^\times$.

Exercise 3.26. Using the description of the points of $\mathbb{D}_K$ given in the previous lecture, check that $|f(x)| \neq 0$ for any $x \in \operatorname{Spa}(A, A^+) \cong \operatorname{Spa}(\widehat{A}, \widehat{A}^+) = \mathbb{D}_{\mathbb{Q}_p}$. For example, given a Type I point $\alpha \in \mathbb{Z}_p$ in Example 2.28, we have $|f(\alpha)|_p = |1 + p\alpha|_p = 1 \neq 0$.

In particular, this implies

$$X = \operatorname{Spa}(A, A^+) = U(1/f).$$

However, there is no map from $A[1/f]$ to A.

3.3.4 Adic affine line

Consider $\mathbb{A}^1 := \operatorname{Spa}(\mathbb{Z}[T], \mathbb{Z})$. If $K, \mathcal{O})$ is a nonarchimedean field, then we have

$$\mathbb{A}^1_K(A, A^+) = A \cong A^+[\pi^{-1}] \cong \operatorname{colim}(A^+ \xrightarrow{\pi} A^+ \xrightarrow{\pi} \cdots)$$

It follows that

$$\begin{aligned}
\mathbb{A}^1_K &\cong \operatorname*{colim}_k(\mathbb{D}_K \xrightarrow{\pi^\#} \mathbb{D}_K \xrightarrow{\pi^\#} \cdots) \\
&\cong \operatorname*{colim}_n \operatorname{Spa}(K\langle \pi^n T\rangle, \mathcal{O}_K\langle \pi^n\rangle),
\end{aligned}$$

i.e. the affine line is the union of closed discs along inclusions of increasing radius.

3.3.5 Adic circle, punctured line, and projective line

We define:

$$\begin{aligned}
\partial\mathbb{D} &:= \operatorname{Spa}(\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{\pm 1}]) \\
\mathbb{G}_m &:= \operatorname{Spa}(\mathbb{Z}[T^{\pm 1}], \mathbb{Z}) \\
\mathbb{P}^1 &:= \mathbb{D} \cup_{\partial\mathbb{D}} \mathbb{D}
\end{aligned}$$

where the gluing for $\mathbb{P}^1$ is $T \mapsto T^{-1}$ [4]. The functors of points are respectively $\partial\mathbb{D}(A, A^+) = (A^+)^\times$, $\mathbb{G}_m(A, A^+) = A^\times$ and $\mathbb{P}^1(A, A^+) = \{[a_0 : a_1] \mid a_i \in A^+\}$. As an example, let's compute the coordinate ring of $\partial\mathbb{D}_K$: we know that

$$\partial\mathbb{D}_K = U(1/f) = \operatorname{Spa}\left(\widehat{K\langle T\rangle\left[\frac{1}{T}\right]}, \widehat{\mathcal{O}_K\langle T\rangle\left[\frac{1}{T}\right]}\right),$$

---

[4] The notation $\mathbb{G}_m$ may be non-standard—we could not find a reference.

and so

$$\begin{aligned}\widehat{K\langle T\rangle}\left[\frac{1}{T}\right] &= \left(\mathcal{O}_K\langle T\rangle\left[\frac{1}{T}\right]\right)^{\wedge}_{(\pi)}\left[\frac{1}{\pi}\right] \\ &= \left\{\sum_{n\in\mathbb{Z}} a_nT^n \,\middle|\, a_n \in \mathcal{O}_K,\ \lim_{|n|\to\infty}|a_n| = 0\right\}\left[\frac{1}{\pi}\right] \\ &= \left\{\sum_{n\in\mathbb{Z}} a_nT^n \,\middle|\, a_n \in K,\ \lim_{|n|\to\infty}|a_n| = 0\right\}.\end{aligned}$$

Another name for this would be $K\langle T^{\pm 1}\rangle$.

3.3.6 Open unit disc

Consider

$$\mathring{\mathbb{D}} := \mathrm{Spa}(\mathbb{Z}[\![T]\!], \mathbb{Z}[\![T]\!]).$$

Then $\mathring{\mathbb{D}}(A, A^+) = A^{\circ\circ}$. We want to describe the points in $\mathring{\mathbb{D}}$. Here are some obvious ones:

- $x_{\mathbb{F}_p}\colon \mathbb{Z}_p[\![T]\!] \to \mathbb{F}_p \to \{0,1\}$, $|p| = |T| = 0$.
- $x_{\mathbb{F}_p(\!(T)\!)}\colon \mathbb{Z}_p[\![T]\!] \to \mathbb{F}_p(\!(T)\!) \to T^{\mathbb{Z}} \cup \{0\}$, $|p| = 0, |T| = T^{-1}$.
- $x_{\mathbb{Q}_p}\colon \mathbb{Z}_p[\![T]\!] \to \mathbb{Q}_p \to p^{\mathbb{Z}} \cup \{0\}$, $|p| = p^{-1}, |T| = 0$.

What other points can we find?

Definition 3.27. A point x in $\mathrm{Spa}(A, A^+)$ is called *analytic* if $\ker x$ is not open.

For example, $x_{\mathbb{F}_p} \in \mathring{\mathbb{D}}_{\mathbb{Z}_p}$ is not analytic, and is the unique non-analytic point. To go further, we use the following lemma:

Lemma 3.28. *Let Γ be a totally ordered abelian group with an element $\gamma \in \Gamma$ such that*

- $0 < \gamma < 1$ *in* $\Gamma \cup \{0\}$*;*
- *for any* $\gamma' \in \Gamma$*, there is an* n *such that* $\gamma^n < \gamma'$*.*

Then any continuous valuation $x\colon (A, A^+) \to \Gamma \cup \{0\}$ *has a* maximal generalization

$$\begin{tikzcd}(A, A^+) \arrow[r, "x"] \arrow[dr, "\tilde{x}"'] & \Gamma \cup \{0\} \arrow[d] \\ & \mathbb{R}_{\geq 0}.\end{tikzcd}$$

in the sense that any other such $\tilde{x}'$ factors uniquely through $\tilde{x}$.

Proof. Pick any $0 < \delta < 1$ in Γ and a real number $r \in (0,1)$ set for any $\gamma \in \Gamma$

$$\varphi_n(\gamma) = r^{m(n)/n}, \qquad \text{where } m(n) = \max\{i \mid \delta^i \geq \gamma^n\}.$$

As the sequence is increasing and bounded, we can define

$$\varphi(\gamma) = \lim_{n\to\infty}\varphi_n(\gamma) \in \mathbb{R}_{\geq 0}.$$

For example, $\varphi_n(\delta^i) = r^i$ for any n and i.

Exercise 3.29. Show that this makes sense, and that $\tilde{x} = \varphi \circ x$ does the job. $\square$

For example, if $x\colon A \twoheadrightarrow \Gamma \cup \{0\}$ is analytic and A is a Huber ring, then there is a $\gamma \in \Gamma$ as in the Lemma. Now suppose $x\colon \mathbb{Z}_p[\![T]\!] \to \Gamma \cup \{0\}$ is an analytic point in $\mathring{\mathbb{D}}$. Then either $|p(x)| \neq 0$ or $|T(x)| \neq 0$, and moreover both $|p(x)|$ and $|T(x)| < 1$ (since p and T are topologically nilpotent). Set

$$\kappa(x) := \frac{\log|T(\tilde{x})|}{\log|p(\tilde{x})|} \in [0,\infty].$$

For example, we have $\kappa(x_{\mathbb{Q}_p}) = \infty$ and $\kappa(x_{\mathbb{F}_p(\!(T)\!)}) = 0$. Observe that

$$\kappa(x) \leq \frac{s}{t} \iff |p(x)|^s \geq |T(x)|^t.$$

Likewise, for any positive real number r, we have $\kappa(x) \leq r$ iff for any $\frac{s}{t} > r$, $|p(x)|^s \geq |T(x)|^t$. The implication still holds with the directions of all inequalities reversed. This property characterizes κ. In this way, we have constructed a map $\kappa\colon \mathring{\mathbb{D}}\backslash\{x_{\mathbb{F}_p}\} \to [0,\infty]$. As $\mathring{\mathbb{D}}\backslash\{x_{\mathbb{F}_p}\} \subseteq \mathring{\mathbb{D}}$ is spectral, hence compact, its image $\kappa(\mathring{\mathbb{D}}\backslash\{x_{\mathbb{F}_p}\})$ is also compact (closed) in $[0,\infty]$.

Exercise 3.30. Prove that $\kappa(\mathring{\mathbb{D}}\backslash\{x_{\mathbb{F}_p}\})$ is dense in $[0,\infty]$ by constructing an x such that $\kappa(x) = \frac{s}{t}$.

Hint: Given a pair of coprime natural numbers $s, t \in \mathbb{N}$, construct:

$$x\colon \mathbb{Z}_p[\![T]\!] \longrightarrow \mathbb{Z}_p[\![T]\!]/(T^t - p^s) \longrightarrow E \longrightarrow \pi^{\mathbb{Z}} \cup \{0\}.$$

Then $|T|^t = |p|^s$ and $\kappa(x) = \frac{s}{t}$.

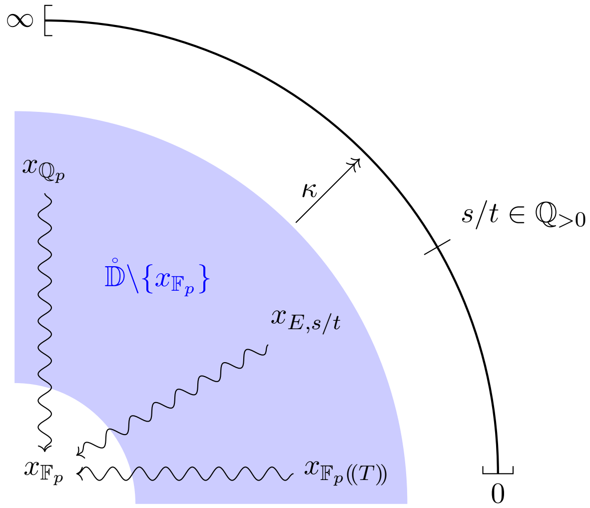

Figure 2: The map $\kappa\colon \mathring{\mathbb{D}}\backslash\{x_{\mathbb{F}_p}\} \to [0,\infty]$

Proposition 3.31. *$\mathring{\mathbb{D}}\backslash\{x_{\mathbb{F}_p}\}$ is an adic space.*

Warning 3.32. $\kappa^{-1}((0,\infty])^\circ$ is *not* affinoid: else by the functor of points, we'd have $\kappa^{-1}((0,\infty])^\circ = \operatorname{Spa}(B, B^+)$ with

$$(B, B^+) = (\mathbb{Z}_p[\![T]\!], \mathbb{Z}_p[\![T]\!]) \otimes_{(\mathbb{Z}_p,\mathbb{Z}_p)} (\mathbb{Q}_p, \mathbb{Z}_p). \tag{3.33}$$

But in that case we'd have $1/p \in B$ and $T \in B^{\circ\circ}$, so $T^n/p \to 0$. In particular $T^N/p \in A^+$ for $N \gg 0$. On the other hand, by the universal property there should be a map $(B, B^+) \to (\mathbb{Q}\langle T, T^{N+1}/p\rangle, \mathbb{Z}_p\langle T, T^{N+1}/p\rangle)$, and this is a contradiction. In particular, the pushout (3.33) does not exist in the category of Huber pairs: the problem is that the map $\mathbb{Z}_p \to \mathbb{Z}_p[\![T]\!]$ is not adic.

Proof (Proposition 3.31). We may cover $\mathring{\mathbb{D}} \setminus \{x_{\mathbb{F}_p}\}$ by rational subsets

$$U(p/T) = \kappa^{-1}([0,1])^\circ$$
$$U(T^n/p) = \kappa^{-1}([1/n,\infty])^\circ$$

for $n \geq 1$: indeed, if $\kappa(x) \neq 0$ then $|p(x)| \neq 0$, so $|T(x)|^N < |p(x)|$ for $N \gg 0$. We'll prove that these are affinoid: that is, we need to show their coordinate rings are sheafy.

1. $(U(p/T))$. We have $U(p/T) = \operatorname{Spa}(B, B^+)$, where
$$B = \mathbb{Z}_p[\![T]\!][p/T]^\wedge_{(p,T)}[T^{-1}] = \mathbb{Z}_p[\![T]\!][p/T]^\wedge_{(T)}[T^{-1}] = \mathbb{Z}_p[\![T]\!]\langle p/T\rangle[T^{-1}].$$
A ring of definition is $\mathbb{Z}[\![T]\!]\langle p/T\rangle$, which is the T-adic completion of $\mathbb{Z}[T, p/T]$ and hence noetherian; thus item (2) of Theorem 3.17 applies.
2. $(U(T^n/p))$. In this case,
$$\mathcal{O}(U(T^n/p)) = \mathbb{Z}_p[\![T]\!][T^n/p]^\wedge_{(p)}[p^{-1}] = \mathbb{Q}_p\langle T, T^n/p\rangle.$$
This is again Tate, and topologically finitely generated over $\mathbb{Z}_p$, which is noetherian. □

Variant 3.34. (A preview of things to come). Suppose that R is a complete Tate ring of characteristic $p > 0$, and moreover R^+ is perfect. In particular, this implies (using the Banach Open Mapping theorem) that R is uniform [SW20, Proposition 6.1.6]. Set $S = \operatorname{Spa}(R, R^+)$.

Definition 3.35. The ring $\mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p}$ is the ring of p-typical Witt vectors,

$$\mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p} \coloneqq W(R^+).$$

Note that we have elements $p, [\varpi] \in \mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p}$, where $[-]$ denotes the multiplicative lift. We equip $\mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p}$ with the $(p, [\varpi])$-adic topology.

Definition 3.36. $\mathcal{Y}_{S,\mathbb{Q}_p} \coloneqq \operatorname{Spa} \mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p} \setminus \{[\varpi] = 0\}$.

Remark 3.37. Note that $\operatorname{Spa} \mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p}$ has a unique non-analytic point

$$x_{\mathrm{na}} : \mathbb{A}_{\mathrm{inf}} \to R^+/\varpi \to \{0,1\}.$$

The subspace $\mathcal{Y}$ looks like a "ϖ-generic fibre" (over S), except that the map $[-]\colon R^+ \to \mathbb{A}_{\mathrm{inf}}$ is not additive and so $\operatorname{Spa} \mathbb{A}_{\mathrm{inf}}$ does not really live over S.

The space $\operatorname{Spa} \mathbb{A}_{\mathrm{inf}}$ looks very similar to $\mathring{\mathbb{D}}_{\mathbb{Z}_p}$: see Fig. 3. In particular, one can define a continuous surjective function $\kappa\colon \operatorname{Spa} \mathbb{A}_{\mathrm{inf}} \setminus \{x_{\mathrm{na}}\} \to [0,\infty]$ just as we did above. The following, which is for example [SW20, Proposition 11.2.1], is the analogue of Proposition 3.31:

Theorem 3.38. *$\mathcal{Y}$ is an adic space.*

The idea is again to cover $\mathcal{Y}$ by rational subsets $U = U(p/[\varpi^{1/p^n}])$, which cover since $|p(x)| < 1$ for any x. Then

$$\mathcal{O}(U) = W(R^+)\left[\frac{p}{[\varpi^{1/p^n}]}\right]^\wedge_{([\varpi])}\left[\frac{1}{[\varpi]}\right].$$

To show that $\mathcal{Y}$ is adic we need to show these rings are sheafy, which is very non-obvious. The strategy will be to construct split coverings of $\mathcal{O}(U)$ by perfectoids, and use this to deduce sheafiness.

Remark 3.39. Scholze and Weinstein remark that it is probably the case that $\operatorname{Spa} \mathbb{A}_{\mathrm{inf}} \setminus \{x_{\mathrm{na}}\}$ is in fact adic too. In any case, note that $\mathcal{Y}$ is not affinoid, since no finite subcover of the $U(p/[\varpi^{1/p^n}])$ covers.

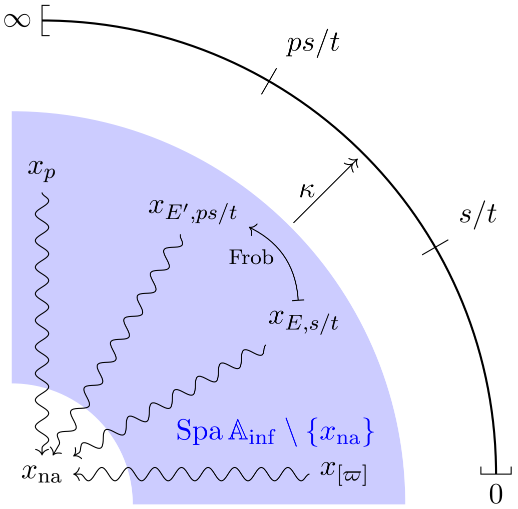

Figure 3: The map $\operatorname{Spa}\mathbb{A}_{\mathrm{inf}} \setminus \{x_{\mathrm{na}}\} \to [0,\infty]$. Here $|p(x_p)| = p^{-1}$ and $|[\varpi](x_p)| = 0$, while $|p(x_{[\varpi]})| = 0$ and $|[\varpi](x_{[\varpi]})| = 0$. In this case the Frobenius on R^+ induces one on $\operatorname{Spa}\mathbb{A}_{\mathrm{inf}}$, and $\kappa(\operatorname{Frob} x) = p\kappa(x)$.

4 The proétale site of an adic space (Emma Brink, 1 July)

4.1 The proétale site

Ultimately our goal in this seminar is to understand the E_2-page of the rationalized $K(h)$-local E_h-Adams spectral sequence,

$$H^*_{\mathrm{cont}}(\mathbb{G}_h, \pi_* E_h),$$

by viewing it as the cohomology of $[\mathrm{LT}_\eta/\mathbb{G}_h]$, the stack quotient of the adic generic fibre of Lubin-Tate space by the Morava stabiliser action. Since $\mathbb{G}_h$ is a profinite group, to make sense of this stack we will view it as a sheaf on the *proétale site* of LT_η, which is what we discuss today. To begin with, we define the relevant sites.

Definition 4.1. A map $X \to Y$ of adic spaces is

1. *analytic* if $|X| \to |Y|$ is an open embedding and $\mathcal{O}_X = \mathcal{O}_Y|_X$.
2. *finite étale* if for any open affinoid $\operatorname{Spa}(A, A^+) \subset Y$, $X \times_Y \operatorname{Spa}(X, X^+) \cong \operatorname{Spa}(B, B^+)$ is affinoid and
    - the map $A \to B$ exhibits B as a finitely generated A-module with induced topology,
    - B^+ is the integral closure of A^+ in B,
    - The A-algebra B has a presentation as $A[X_1, \dots, X_n]/(f_1, \dots, f_n)$ with $\det(\partial f_i/\partial X_j)_{ij} \neq 0$.
3. *étale* if there is an open covering $X = \bigcup_i U_i$ and affinoid opens $f(U_i) \subset V_i \subset_o Y$ such that each $f|_{U_i} \colon U_i \to V_i$ is finite étale.
4. *proétale* if there exists a covering $X = \bigcup_i U_i$ and $f(U_i) \subset V_i \subset_o Y$ such that
    - $U_i = \varprojlim_{J_i} \operatorname{Spa}(A_{ij}, A_{ij}^+)$ is the cofiltered limit of affinoids and
    - $f|_{U_i} \colon U_i \to V_i$ is the limit of étale maps $f_{ij} \colon \operatorname{Spa}(A_{ij}, A_{ij}^+) \to V_i$.

Definition 4.2. A family of morphisms of adic spaces $\{f_i \colon X_i \to Y\}_I$ is an *qc cover* if for all quasicompact opens $U \subset Y$, there exists a quasicompact open $U \subset \coprod_{i \in I} X_i$ such that $U \subset f(C)$.

Definition 4.3. For an adic space X, we define the following sites:

1. $X_{\mathrm{Zar}} = X_{\mathrm{an}}$ is the category of adic spaces equipped with an analytic map to X, with the topology given by analytic qc covers.
2. $X_{\text{ét}}$ is the category of adic spaces equipped with an étale map to X, with the topology given by étale qc covers.
3. $X_{\text{proét}}$ is the category of adic spaces equipped with a proétale map to X, with the topology given by proétale qc covers.

In particular there are obvious inclusions $X_{\mathrm{Zar}} \xrightarrow{\iota} X_{\text{ét}} \xrightarrow{\nu} X_{\text{proét}}$, which give rise to geometric morphisms

$$\mathrm{Sh}(X_{\mathrm{Zar}}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathrm{Sh}(X_{\text{ét}}) \underset{\nu_*}{\overset{\nu^*}{\rightleftarrows}} \mathrm{Sh}(X_{\text{proét}})$$

on the associated 1- or ∞-topoi.

Remark 4.4. If X is affinoid then it is essentially by definition that $X_{\text{ét}}$ is generated under colimits by the subcategory $X_{\text{ét}}^{\mathrm{aff}}$ of *affinoids* étale over X; likewise, $X_{\text{proét}}$ is generated under colimits by the subcategory $X_{\text{proét}}^{\mathrm{aff}}$, and there is an equivalence

$$\varprojlim : \mathrm{Pro}(X_{\text{ét}}^{\mathrm{aff}}) \xrightarrow{\sim} X_{\text{proét}}^{\mathrm{aff}}.$$

As a warning, this is no longer true when we take X to be a general adic space, since we might not be able to glue pro-affinoid presentations.

4.2 The case of a point

We will consider in detail the case that $X = \mathrm{Spa}(K, K^\circ)$ is the adic spectrum of a nonarchimedean field, in which case

$$X \simeq \mathrm{Cont}(K) \simeq \mathrm{Spv}(K^\circ/K^{\circ\circ}),$$

which is a point when $K^\circ/K^{\circ\circ}$ is finite (e.g. when $K = \mathbb{Q}_p$). See [Mor19] for the details of these equivalences. Write $G = G_K := \mathrm{Gal}(K^s/K)$ for the Galois group of the separable closure K^s of K.

1. $X_{\mathrm{Zar}} = \{\emptyset, *\}$, hence a sheaf on X_{Zar} is just an abelian group.
2. If $K \subseteq L$ is a finite separable field extension, denote by $L^+ := \overline{K}^L$ the integral closure of K in L and equip $L \cong K^n$ with the topology induced from K. Then the map $\mathrm{Spa}_K(L) := \mathrm{Spa}(L, L^+) \to \mathrm{Spa}(K, K^+)$ induced by $L \hookrightarrow K$ is étale. Every affinoid adic space étale *over* $\mathrm{Spa}(K, K^+)$-scheme is a coproduct $\sqcup_{i\in I} \mathrm{Spa}_K(L_i, L_i^+)$ for finite separable field extensions $K \subseteq L_i$. Denote by Set_G the category of discrete G-sets with a continuous G-action and G-equivariant maps.

   Sending an orbit G/H for a finite-index subgroup $H \subseteq G$ to $\mathrm{Spa}_K((K^s)^H)$ defines an equivalence
   $$\mathrm{Set}_G \xrightarrow{\sim} X_{\text{ét}}^{\mathrm{aff}}.$$

   Thus,
   $$\begin{aligned} \mathrm{Sh}(X_{\text{ét}}) &\simeq \mathrm{Sh}(X_{\text{ét}}^{\mathrm{aff}}) \\ &\simeq \mathrm{Sh}(\mathrm{Set}_G). \end{aligned}$$

   The topology on Set_G is generated by jointly surjective families of maps. Via restriction, the right hand side is further equivalent to sheaves on the site of finite G-sets with jointly surjective covers, which are automatically refined by a finite subfamily.[5] In particular, this implies that cohomology of étale sheaves agrees with continuous group cohomology of *discrete* G-modules.

3. By the same reasoning,

$$\begin{aligned}\mathrm{Sh}(X_{\text{proét}}) &\simeq \mathrm{Sh}(X^{\text{aff}}_{\text{proét}}) \\ &\simeq \mathrm{Sh}(\mathrm{Pro}(\mathrm{Fin}_G)) \\ &\simeq \mathrm{Sh}(\mathrm{Profin}_G) \\ &\simeq \mathrm{Mod}_{\underline{G}}(\mathrm{Cond}(\mathrm{Set}))\end{aligned}$$

where Profin_G denotes the category of profinite sets with continuous G-action, and the topology in the second and third lines is given in both cases by families refined by some finite surjective subfamily.

The equivalence $\mathrm{Sh}(\mathrm{Pro}(\mathrm{Fin}_G)) \simeq \mathrm{Sh}(\mathrm{Profin}_G)$ holds since

$$\varprojlim : \mathrm{Pro}(\mathrm{Fin}_G) \to \mathrm{Profin}_G$$

is fully faithful and its image generates Profin_G under colimits.

Remark 4.5. The final equivalence follows from the following general fact: if G is a group object in a site $\mathcal{C}$ (i.e. a group object in the underlying category), then $\mathrm{Sh}(B_{\mathcal{C}}G) \simeq B_{\mathrm{Sh}(\mathcal{C})}G$.

We want to relate sheaf cohomology on $X_{\text{proét}}$ to continuous group cohomology. Suppose that M is a continuous locally profinite G-module, or more generally any $T1$ topological G-module for which the associated condensed abelian group $\underline{M}$ is solid. We will show that

$$H^*(X_{\text{proét}}, M) \cong H^*_{\text{cont}}(G, M).$$

The cover $\{\underline{G} \to *\} \in \mathcal{B}_{\mathrm{Cond(Set)}}\underline{G}$ yields a Čech-to-cohomology spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathrm{Ab}(\mathcal{B}_{\mathrm{Cond(Set)}}\underline{G})}(\mathbb{Z}[\underline{G}^{p+1}], -) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{Ab}(\mathcal{B}_{\mathrm{Cond(Set)}})\underline{G}}(\mathbb{Z}, -) = H^*(X_{\text{proét}}, -).$$

Denote by S^G_* the simplicial complex associated with the Čech nerve of $\{G \to *\}$. If M is a condensed $\mathbb{Z}[\underline{G}]$-module with

$$\mathrm{Ext}^j_{\mathrm{Cond}(\mathbb{Z}[\underline{G}]}(\mathbb{Z}[\underline{G}^i], \underline{M}) = 0 \text{ for } i, j \in \mathbb{N}_1,$$

then the Čech-to-cohomology spectral sequence collapses and yields

$$H^*(\mathrm{Hom}_{\mathbb{Z}[\underline{G}]}(\mathbb{Z}[\underline{G}^*], M)) = \mathrm{Ext}^*_{\mathcal{B}_{\underline{G}}(\mathrm{Cond(Set)})}(\mathbb{Z}, M) = H^*(X_{\text{proét}}, M).$$

As G^i is compactly generated for all $i \in \mathbb{N}_0$, for a $T1$ continuous G-module M,

$$\mathrm{Hom}_{\mathbb{Z}[\underline{G}]}(\underline{S^G_*}, \underline{M}) \cong \mathrm{Cont}(G^{\times *}, M)$$

is the complex of continuous cochains, which by definition computes the continuous group cohomology.

For a condensed $\mathbb{Z}[\underline{G}]$-module M and $i \in \mathbb{N}_1$,

$$\mathrm{Ext}^j_{\mathrm{Cond}(\mathbb{Z}[\underline{G}]}(\mathbb{Z}[\underline{G}^i], M) \cong \mathrm{Ext}^j_{\mathrm{Cond(Ab))}}(\mathbb{Z}[G^{i-1}], M).$$

---

[5] Here the following fact is used: if $\mathcal{D}$ is a site and $\mathcal{C}$ a full subcategory closed under pullbacks, equipped with a topology $\tau_{\mathcal{C}} \subset \tau_{\mathcal{D}}$ and generating in the sense that any covering in $\mathcal{D}$ is refined by one in $\mathcal{C}$, then restriction induces an equivalence $\mathrm{Sh}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Sh}(\mathcal{C})$.

If G is not discrete, then $\mathbb{Z}[G^2]$ is not projective, so $\mathrm{Ext}^1_{\mathrm{Cond}(\mathbb{Z}[\underline{G}]}(\mathbb{Z}[\underline{G}^i], -) \neq 0$.

But for solid abelian groups M, $\mathrm{Ext}^j_{\mathrm{Cond(Ab))}}(\mathbb{Z}[G^{i-1}], M) = \mathrm{Ext}^j_{\mathrm{Solid(Ab)}}(\mathbb{Z}[G^{i-1}]^{L\square}, M)$ and we have:

Proposition 4.6 ([Sch19], Theorem 5.8)**.** *For any profinite space X, we have that*

$$\mathbb{Z}[\underline{X}]^{L\square} \simeq \mathbb{Z}[\underline{X}]^{\square}$$

is concentrated in degree zero, and this is a projective object of Solid(Ab)*.*

In particular, for a $T1$ continuous G-module M with $\underline{M}$ solid, (e.g. M locally profinite),

$$\mathrm{Ext}^j_{\mathrm{Cond}(\mathbb{Z}[\underline{G}]}(\mathbb{Z}[\underline{G}^i], \underline{M}) = 0 \text{ for } i, j \in \mathbb{N}_1,$$

whence

$$H^*(X_{\text{proét}}, M) \cong H^*_{cont}(G, M).$$

The idea of proof for the proposition is to use the fact, due to Nöbeling and Specker, that $C(X, \mathbb{Z})$ is a free abelian group for any profinite set X, and that $\mathbb{Z}[\underline{X}]^{\square} = \underline{\mathrm{Hom}}_{\mathrm{Cond(Ab)}}(C(X, \mathbb{Z}), \mathbb{Z})$.

All in all, we obtain for any locally profinite G-module M, maps

$$H^*(X_{\mathrm{Zar}}, M) \to H^*(X_{\text{ét}}, M) \to H^*(X_{\text{proét}}, M),$$

where moreover

$$\begin{aligned} H^*(X_{\mathrm{Zar}}, M) &\cong \begin{cases} M^G & * = 0 \\ 0 & * \neq 0 \end{cases} \\ H^*(X_{\text{ét}}, M) &\cong H^*_{cont}(G, M^{\delta}) := \varinjlim H^*(G/U, M^U) \\ H^*(X_{\text{proét}}, M) &\cong H^*_{cont}(G, M) \end{aligned}$$

where the colimit in the second line runs over all open normal subgroups of G.

Remark 4.7. In general, for a condensed group $\mathcal{G}$ and a condensed $\mathcal{G}$-module $M \in \mathrm{Ab}(\mathcal{B}_{\mathrm{Cond(Set)}}\mathcal{G})$, one has a Čech-to-cohomology spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathrm{Ab}(\mathcal{B}_{\mathrm{Cond(Set)}}\mathcal{G})}(\mathbb{Z}[\mathcal{G}^{p+1}], -) = \mathrm{Ext}^q_{\mathrm{Cond(Ab)}}(\mathbb{Z}[\mathcal{G}^p], -) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{Ab}(\mathcal{B}_{\mathrm{Cond(Set)}}\mathcal{G})}(\mathbb{Z}, -)$$

which converges to the condensed group cohomology of $\mathcal{G}$. Condensed group cohomology is therefore in general a finer invariant than continuous group cohomology which ignores all terms $E_2^{p,q}$ for $p, q > 1$ in the above spectral sequence. For example, one can show that for a $T1$ topological abelian group G and a locally profinite abelian group M with trivial $\underline{G}$-action,

$$\mathrm{Ext}^*_{\mathrm{Ab}(\mathcal{B}_{\mathrm{Cond(Set)}}\mathcal{G})}(\mathbb{Z}, \underline{M}) = \mathrm{Ext}^{p+q}_{\mathrm{Cond(Ab)}}(\mathbb{Z}[\underline{BG}], -)$$

is the condensed cohomology of the classifying space BG of G, whereas

$$H^*_{cont}(G, M) = H^*_{cont}(\pi_0 G, M) = \mathrm{Ext}^{p+q}_{\mathrm{Cond(Ab)}}(\mathbb{Z}[\underline{B\pi_0 G}], M)$$

is the condensed cohomology of the classifying space $B\pi_0 G$. However, Proposition 4.6 implies that continuous group cohomology with solid coefficients can (for large classes of groups) be realised as Ext-groups in the condensed world.

Denote by $\mathrm{Solid}(\mathbb{Z}[\underline{G}]) \subseteq \mathrm{Cond}(\mathbb{Z}[\underline{G}])$ the full subcategory on condensed $\mathbb{Z}[\underline{G}]$-modules whose underlying condensed abelian group is solid. This is an abelian subcategory closed under limits.

The inclusion $\mathrm{Solid}(\mathbb{Z}[\underline{G}]) \subseteq \mathrm{Cond}(\mathbb{Z}[\underline{G}])$ has a left adjoint $(-)^{\square G}$ which sends a $\mathbb{Z}[\underline{G}]$-module to its solidification with the induced $\mathbb{Z}[\underline{G}]$-module structure. Solidification is not an exact functor, but one can show that for a $T1$ topological group, the degreewise solidificaion $(S_*^{\underline{G}})^{\square G}$ of the simplicial resolution is a resolution of $\mathbb{Z} \in \mathrm{Solid}(\mathbb{Z}[\underline{G}])$. If this is a projective resolution and G^i is compactly generated for all $i \in \mathbb{N}_0$), we obtain that

$$\mathrm{Ext}^*_{\mathrm{Solid}(\mathbb{Z}[\underline{G}])}(\mathbb{Z}, \underline{-}) \cong H^*_{cont}(G, -)$$

on continuous G-modules whose associated condensed abelian group is solid, like locally profinite continuous G-modules.

Since solid tensor products of projectives in $\mathrm{Solid}(\mathrm{Ab})$ are projective, $(S_*^{\underline{G}})^{\square G}$ is a projective resolution of $\mathbb{Z}$ in $\mathrm{Solid}(\mathbb{Z}[\underline{G}])$ if and only if $\mathbb{Z}[\underline{G}]^{\square}$ is a projective solid abelian group. This holds in many cases, for example if G is locally connected or if G is a coproduct of compact spaces or a product of two such groups. It works in slightly larger generality, and we currently do not know an example where it fails. It might fail for groups which are totally disconnected but not locally compact like the rationals with euclidean topology.

4.3 Cohomology of $\hat{\mathcal{O}}_X$

We now return to the case of a general affinoid X. There is a functor

$$\begin{aligned} X_{\text{ét}}^{\text{aff}} &\to \mathrm{Top\,Ring} \\ (A, A^+) &\mapsto A^+ \end{aligned}$$

and this satisfies étale descent. We obtain a sheaf of topological rings $\mathcal{O}^+_{X_{\text{ét}}}$ on $X_{\text{ét}}$, and hence a sheaf of topological rings

$$\mathcal{O}^+ := \nu^* \mathcal{O}^+_{X_{\text{ét}}} \in \mathrm{Sh}(X_{\text{proét}}; \mathrm{Top}(\mathrm{Ring})).$$

Denote by $\hat{\mathcal{O}}^+ := \varprojlim_n \mathcal{O}^+/p^n \in \mathrm{Sh}(X_{\text{proét}}; \mathrm{Top}(\mathrm{Ring}))$ the p-completion of $\mathcal{O}^+$, i.e. the sheafification of $T \mapsto \varprojlim_n \mathcal{O}^+(T)/p^n$.

By pointwise passing to condensed sets, we obtain sheaves of condensed rings $\underline{\mathcal{O}^+}$ and $\underline{\hat{\mathcal{O}}^+}$. Since $\underline{-}\colon \mathrm{Top} \to \mathrm{Cond}(\mathrm{Set})$ is a right adjoint, no further sheafification is necessary, i.e.

$$\underline{\mathcal{O}^+} = \underline{-} \circ \mathcal{O}^+ \text{ and } \underline{\hat{\mathcal{O}}^+} = \underline{-} \circ \hat{\mathcal{O}}^+.$$

Example 4.8. If $Y = \mathrm{Spa}(R, R^+) = [\varprojlim_{i \in I} \mathrm{Spa}(Y_i, Y_i^+)]_p^\wedge$ is perfectoid affinoid,

$$\hat{\mathcal{O}}^+(Y) = R^+ = [\varinjlim_{i \in I} Y_i^+]_p^\wedge = [\varinjlim_{i \in I} \hat{\mathcal{O}}^+(Y_i)]_p^\wedge \tag{4.9}$$

is the p-completion of the topological ring $\varinjlim_{i \in I} Y_i^+$.

In the rest of the talk, we discuss:

Proposition 4.10 ([Bar+24], Lemma 3.7.1)**.** $\mathbf{R}\Gamma(X_{\text{proét}}, \underline{\hat{\mathcal{O}}}^+) \simeq \left[\underline{\mathbf{R}\Gamma(X_{\text{proét}}, \mathcal{O}^+_\delta)}\right]_p^\wedge$ *for X affinoid perfectoid, where $\mathcal{O}^+_\delta$) denotes the sheaf of discrete rings underlying $\mathcal{O}^+$.*

Remark 4.11. The functor

$$\begin{aligned} j : *_{\text{proét}} &\to X_{\text{proét}} \\ \varprojlim S_i &\mapsto \varprojlim (S_i \times X) \end{aligned}$$

induces an adjunction

$$j^* : \mathrm{Cond(Set)} \rightleftarrows \mathrm{Sh}(X_{\text{proét}}) : j_*.$$

For every sheaf of abelian groups $F \in \mathrm{Sh}(X_{\text{proét}}, \mathrm{Ab})$, $\mathbf{R}j_*F \in \mathcal{D}(\mathrm{Cond(Ab)})$ has $\mathbf{R}\Gamma(X_{\text{proét}}, F)$ as underlying complex of abelian groups.

But as we will see below, this yields no ambiguity in the condensed structure:

Lemma 4.12. *For any affinoid X there is a natural equivalence*

$$\mathbf{R}j_*\hat{\mathcal{O}}_\delta{}^+ \simeq \mathbf{R}\Gamma(X_{\text{proét}}, \underline{\hat{\mathcal{O}}}^+).$$

The proof of this lemma requires some preparation, see below.

Definition 4.13. An adic space X is *strictly totally disconnected* (abbreviated to *std*) if for any étale cover $\{X_i \to X\}$ there is a finite subset $F \subset I$ such that

$$\coprod_{i \in F} X_i \to X$$

admtits a section. In particular, this implies that $\Gamma\colon \mathrm{Sh}(X_{\text{ét}}; \mathcal{A}) \to \mathcal{A}$ is exact for any abelian $\mathcal{A}$.

Lemma 4.14. *For any std perfectoid affinoid Y, $\mathbf{R}\Gamma(Y_{\text{proét}}, \hat{\mathcal{O}}^+) = \hat{\mathcal{O}}^+(Y)$ and $\mathbf{R}j_*\hat{\mathcal{O}}^+ = j_*\hat{\mathcal{O}}^+$ are concentrated in degree zero.*

Proof. Let $\Gamma_{\text{cond}}\colon \mathrm{Cond(Ab)} \to \mathrm{Ab}$ denote the global section/evaluation at $*$. This is an exact functor and in particular induces a functor $\mathbf{R}\Gamma_{\text{cond}}\colon \mathcal{D}(\mathrm{Cond(Ab)}) \to \mathcal{D}(\mathrm{Ab})$, given by *componentwise application.* It suffices to show that $\mathbf{R}\Gamma_{\text{cond}}\mathbf{R}\Gamma(Y_{\text{proét}}, \hat{\mathcal{O}}^+)$ and $\mathbf{R}\Gamma_{\text{cond}}(\mathbf{R}j_*\hat{\mathcal{O}}^+)$ are concentrated in degree 0.

Now

$$\mathbf{R}\Gamma_{\text{cond}} \circ \mathbf{R}j_* = \mathbf{R}(\Gamma_{\text{cond}} \circ j_*) = \mathbf{R}\Gamma(Y_{\text{proét}}, -)$$

and

$$\mathbf{R}\Gamma_{\text{cond}} \circ \mathbf{R}\Gamma(Y_{\text{proét}}, -) = \mathbf{R}(\Gamma_{\text{cond}} \circ \Gamma(Y_{\text{proét}}, -)) = \mathbf{R}\Gamma(Y_{\text{proét}}, -)$$

as functors $\mathrm{Sh}(Y_{\text{proét}}; \mathrm{Ab}) \to \mathcal{D}(\mathrm{Ab})$. So it suffices to show that $\mathbf{R}\Gamma(Y_{\text{proét}}, \hat{\mathcal{O}}^+)$ is concentrated in degree 0.

By [BS15, Corollary 5.16], the sheafification

$$\nu^*\colon \mathrm{Sh}(Y_{\text{ét}}; \mathrm{Ab}) \to \mathrm{Sh}(Y_{\text{proét}}; \mathrm{Ab})$$

is fully faithful and for any étale sheaf F,

$$\mathbf{R}\Gamma(Y_{\text{proét}}, \nu^*F) = \mathbf{R}\Gamma(Y_{\text{ét}}, F) = F(Y)$$

is concentrated in degree 0 as Y is strictly totally disconnected.

For all $m \in \mathbb{N}_0$ we have an exact sequence

$$0 \to \lim{}^1 H^{m-1}(Y_{\text{proét}}, \hat{\mathcal{O}}^+/p^n) \to H^m(Y_{\text{proét}}, R\lim \hat{\mathcal{O}}^+/p^n) \to \varprojlim_n H^m(Y_{\text{proét}}, \hat{\mathcal{O}}^+/p^n) \to 0.$$

By definition of perfectoid rings, for all affinoid perfectoid Y, $\hat{\mathcal{O}}^+(Y) = R\lim \hat{\mathcal{O}}^+/p^n(Y)$ is derived p-complete. As we will see next week, perfectoid affinoids generate the topology on

$X_{\text{proét}}$, hence $R\lim \hat{\mathcal{O}}^+/p^n = \hat{\mathcal{O}}^+$ as sheaves on $X_{\text{proét}}$. Since X is affinoid perfectoid, for all $n \in \mathbb{N}_1$, $\hat{\mathcal{O}}^+/p^n = \nu^* O^+/p^n$. Hence, for $m \in \mathbb{N}_1$:

$$\varprojlim_n H^m(Y_{\text{proét}}, \hat{\mathcal{O}}^+/p^n) = \varprojlim_n H^m(Y_{\text{ét}}, O^+/p^n) = 0$$

and for $m \in \mathbb{N}_2$,

$$\lim{}^1 H^{m-1}(Y_{\text{proét}}, \hat{\mathcal{O}}^+/p^n) = \lim{}^1 H^{m-1}(Y_{\text{ét}}, O^+/p^n) = 0.$$

This implies that $H^m(Y_{\text{proét}}, \hat{\mathcal{O}}^+) = H^{m-1}(Y_{\text{proét}}, R\lim \hat{\mathcal{O}}^+/p^n) = 0$ for $m \geq 2$ and

$$H^1(Y_{\text{proét}}, \hat{\mathcal{O}}^+) = \lim{}^1 H^0(Y_{\text{proét}}, \hat{\mathcal{O}}^+/p^n) = \lim{}^1 R^+(Y)/p^n = 0$$

since $\hat{\mathcal{O}}^+(Y) = R^+(Y)$ is derived p-complete. This shows that $\mathbf{R}\Gamma(Y_{\text{proét}}, \hat{\mathcal{O}}^+)$ is concentrated in degree 0. □

We will see next week that any X admits a proétale covering by strongly totally disconnected perfectoid affinoids. We may therefore compute proétale cohomology of perfectoid affinoid spaces using any hypercovers by std perfectoid affinoids. This now implies Lemma 4.12:

Proof (Lemma 4.12). Since every rigid analytic space X admits a proétale covering by strongly totally disconnected perfectoid affinoids, we can compute $\mathbf{R}\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+)$ and $\mathbf{R}j_* \hat{\mathcal{O}}^+$ using a hypercover by strongly totally disconnected perfectoid affinoids. Whence it suffices to show that for totally disconnected affinoid perfectoid Y, $\mathbf{R}\Gamma(Y_{\text{proét}}, \hat{\mathcal{O}}^+) \cong \mathbf{R}j_* \hat{\mathcal{O}}^+$ naturally with respect to pro-étale morphisms.

By Lemma 4.14, for Y totally disconnected affinoid perfectoid, $\mathbf{R}\Gamma(Y_{\text{proét}}, \underline{\hat{\mathcal{O}}^+}) \cong \underline{\hat{\mathcal{O}}^+(Y)}$ and $\mathbf{R}j_* \hat{\mathcal{O}}^+ = j_* \hat{\mathcal{O}}^+$ are concentrated in degree 0, so we are reduced to showing that for Y strongly totally disconnected affinoid perfectoid, $j_* \hat{\mathcal{O}}^+(Y) \cong \Gamma(Y_{\text{proét}}, \underline{\hat{\mathcal{O}}}^+)$, naturally with respect to pro-étale morphisms. We will show that this holds for all affinoid perfectoid spaces Y.

For $S = \lim_{i \in I} S_i \in \text{Pro(Fin)}$ and $X = \text{Spa}(R, R^+)$ perfectoid affinoid,

$$X \times S := [\varprojlim_{i \in I} S_i \times Y]^\wedge_p = [\varprojlim_{i \in I} \text{Spa}(R^{S_i}, (R^+)^{S_i})]^\wedge_p$$

is perfectoid affinoid and $X \times S \in X_{\text{proét}}$.

Hence, by definition of j_*,

$$\begin{aligned}
j_* \hat{\mathcal{O}}^+(\varprojlim_{i \in I} S_i) &= \text{Hom}_{X_{\text{proét}}}(\varprojlim_{i \in I} X \times S_i, \hat{\mathcal{O}}^+) \\
&= \hat{\mathcal{O}}^+(\varprojlim_{i \in I} X \times S_i) \\
&\overset{4.8}{=} [\varinjlim_{i \in I} \hat{\mathcal{O}}^+(X \times S_i)]^\wedge_p \\
&= [\varinjlim_{i \in I} (\hat{\mathcal{O}}^+(X))^{S_i}]^\wedge_p \\
&= [\varinjlim_{i \in I} \mathcal{C}(S_i, \hat{\mathcal{O}}^+(X))]^\wedge_p \\
&= [\varinjlim_{i \in I} \mathcal{C}(S_i, R^+(X))]^\wedge_p.
\end{aligned}$$

Since $\hat{\mathcal{O}}^+(X) = R^+ = \varprojlim_n R^+/p^n$ is profinite and R^+/p^n is finite discrete for all $n \in \mathbb{N}_1$ by definition of perfectoid rings,

$$\underline{R^+}(\varprojlim_{i\in I} S_i) = \mathcal{C}(\varprojlim_{i\in I} S_i, R^+) = \mathcal{C}(\varprojlim_{i\in I} S_i, \varprojlim_{n} R^+/p^n) = \varprojlim_{n} \varinjlim_{i} \mathcal{C}(S_i, R^+/p^n).$$

As for all finite sets F and $n \in \mathbb{N}$, $C(F, R^+/p^n) = \mathcal{C}(F, R^+)/p^n$,

$$\varprojlim_{n} \varinjlim_{i} \mathcal{C}(S_i, R^+/p^n) = \varprojlim_{n} \varinjlim_{i} (\mathcal{C}(S_i, R^+)/p^n) = \varprojlim_{n} (\varinjlim_{i} \mathcal{C}(S_i, R))/p^n = [\varinjlim_{i\in I} (\mathcal{C}(S_i, R^+)]^{\wedge}_p.$$

This shows that

$$\underline{R^+}(\varprojlim_{i\in I} S_i) \cong [\varinjlim_{i\in I} (\mathcal{C}(S_i, R^+)]^{\wedge}_p \cong j_* \hat{\mathcal{O}}^+(\varprojlim_{i\in I} S_i)$$

for all $S = \varprojlim_{i\in I} S_i \in \mathrm{Pro(Fin)}$. These identifications obviously define isomorphism of condensed sets/abelian groups/rings and are natural in the perfectoid affinoid space X with respect to pro-étale morphisms. $\square$

In fact, we do not need complicated hypercovers but can even work with Čech nerves of covers to compute $\mathbf{R}\Gamma(Y, \underline{\hat{\mathcal{O}}}^+)$:

Lemma 4.15 ([Bar+24], Lemmas 3.7.3-4)**.** *Let Y be perfectoid affinoid. There exists a proétale cover $X \to Y$ with X std perfectoid affinoid. All terms $X^{(i+1)} := X^{\times_Y i}$ in the Čech nerve are also std perfectoid affinoid.*

Proof (Proposition 4.10). Given this, we obtain the desired identification

$$\begin{aligned}
\mathbf{R}\Gamma(Y, \underline{\hat{\mathcal{O}}}^+) &\simeq \mathrm{Tot}\, \underline{\hat{\mathcal{O}}}^+(X^{(\bullet)}) \\
&\simeq \mathrm{Tot} \varprojlim_{n} (\underline{\mathcal{O}^+}(X^{(\bullet)})/p^n) \\
&\simeq \mathrm{Tot} \varprojlim_{n} (\underline{\mathcal{O}^+_\delta(X^{(\bullet)})}/p^n) \\
&\simeq \varprojlim_{n} \mathrm{Tot}(\underline{\mathcal{O}^+_\delta(X^{(\bullet)})})/p^n \\
&\simeq \Big[(\mathrm{Tot}\, \underline{\mathcal{O}^+_\delta}(X^{(\bullet)}))\Big]^{\wedge}_p \\
&\simeq \Big[\mathbf{R}\Gamma(Y_{\text{proét}}, \underline{\mathcal{O}^+_\delta})\Big]^{\wedge}_p \\
&\simeq \Big[\underline{\mathbf{R}\Gamma(Y_{\text{proét}}, \mathcal{O}^+_\delta)}\Big]^{\wedge}_p .
\end{aligned}$$

where $\mathcal{O}^+_\delta$ denotes the sheaf of discrete rings underlying $\hat{\mathcal{O}}^+$.

Here we used that for X perfectoid affinoid, $\hat{\mathcal{O}}^+(X) = [\mathcal{O}^+(X)]^{\wedge}_p = \lim_n \mathcal{O}^+(X)/p^n$ and $\mathcal{O}^+(X)/p^n$ is discrete for all $n \in \mathbb{N}_0$, whence

$$\underline{\hat{\mathcal{O}}^+(X)} = \lim_{n} \underline{\mathcal{O}^+_\delta(X)}/p^n. \qquad \square$$

5 Perfectoid spaces (Christian Kremer, 8 July)

5.1 History

Let us recall a classical theorem that admits a generalisation in the language of perfectoid spaces.

Theorem 5.1 (Fontaine–Wintenberger)**.** *There is an isomorphism of absolute Galois groups*

$$\mathrm{Gal}(\mathbb{Q}_p[p^{1/p^\infty}]) \cong \mathrm{Gal}(\mathbb{F}_p(\!(t)\!)[t^{1/p^\infty}]).$$

While these two fields are radically different, we can write any element of $\mathbb{Q}_p$ as a Laurent series in the variable p with coefficients in $\{0, \ldots, p-1\}$. The same thing is true for $\mathbb{F}_p(\!(t)\!)$ per construction, replacing p by the variable t, and we will see that the adjunction of sufficiently many p-th roots of this variable allows us to obtain an isomorphism on absolute Galois groups.

5.2 Perfectoid rings

Let us fix a prime p throughout the rest of this talk, and Let R be a Tate ring, i.e. a Huber ring such that one can choose a topologically nilpotent unit, called the pseudo-uniformiser ϖ. As usual, let $R^\circ \subset R$ denote the subring of power-bounded elements.

Definition 5.2. A Tate ring A is perfectoid if it is complete, uniform (i.e. $A^\circ \subset A$ is a bounded subset), and one can choose a pseudouniformiser ϖ such that

1. $\varpi^p \mid p$
2. The Frobenius map $\mathrm{Frob}\colon A^\circ/\varpi \to A^\circ/\varpi^p$ is an isomorphism.

Remark 5.3. The first condition $\varpi^p \mid p$ implies that A°/ϖ is of characteristic p so the Frobenius map is a ring map.

Remark 5.4. If A is of characteristic p, the conditions $\varpi^p \mid p$ does not make sense prima facie, but we simply set it to be true.

Lemma 5.5. *Let A be a Tate ring with a pseudo-uniformiser ϖ such that $\varpi^p \mid p$. Then*

1. *The Frobenius map $\mathrm{Frob}\colon A^\circ/\varpi \to A^\circ/\varpi^p$ is injective.*
2. *If A is complete and uniform, then*

$$\mathrm{Frob}\colon A^\circ \to A^\circ/\varpi^p$$

is surjective if and only if

$$\mathrm{Frob}\colon A^\circ \to A^\circ/p$$

is surjective.

An upshot of the second part of this lemma is that if a Tate ring is complete and uniform and we already have a pseudo-uniformiser ϖ such that $\varpi^p \mid p$, then the final condition in asking for a certain Frobenius map to be an isomorphism does not depend on the choice of ϖ. Furthermore, the map that we want to be an isomorphism is always injective so it suffices to check it is surjective.

Proof.

1. Suppose $a \in A^\circ$ is (such that its reduction modulo ϖ is) in the kernel of the Frobenius map $A^\circ/\varpi \to A^\circ/\varpi^p$, i.e. there exists a y in A° such that $a^p = y\varpi^p$. Then we have
$$\frac{a^p}{\varpi^p} = y \in A^\circ.$$
By the definition of power-bounded elements, we see that a/ϖ is also contained in A°, whence $a = 0$ in A°/ϖ.
2. Since $\varpi^p \mid p$ by assumption, there is a commutative diagram
$$\begin{tikzcd}
A^\circ \arrow[r, "\phi_1"] \arrow[dr, "\phi_2"'] & A^\circ/p \arrow[d, two heads] \\
 & A^\circ/\varpi^p,
\end{tikzcd}$$
where the vertical map is reduction, hence clearly surjective. It is therefore clear that ϕ_1 being surjective implies that ϕ_2 is surjective, which proves one direction of the statement. For the converse, assume that ϕ_2 is surjective. Let $a \in A^\circ$ be an element of which we want to find a lift along the Frobenius. Since ϕ_2 is surjective, we can write
$$a = a_0^p + b_0\varpi^p.$$
Let us iterate this process for b_0 obtain a power series expansion for a in the variable ϖ^p. Since A is assumed to be Tate, complete, and uniform, A° will be ϖ-adically complete so that the power series expansion above converges, and we write
$$a = \sum_{n\geq 0} a_n^p \varpi^{np}.$$
If we now define
$$c = \sum_{n\geq 0} a_n \varpi^n \in A^\circ,$$
then it is clear by the binomial formula and the fact that $\varpi^p \mid p$ that
$$c^p = a + p(\cdots)$$
whence c is a lift of the class of a along ϕ_1. ☐

Let us now relate the notion of perfectoid rings in characteristic p to the notion of perfect $\mathbb{F}_p$-algebras (we will see that they are quite related, but still quite different).

Proposition 5.6. *Let A be Tate of characteristic p, then the following are equivalent.*

1. *A is perfectoid.*
2. *A is complete and perfect.*

Remark 5.7. Note that this proposition applies to Tate algebras, in particular while $\mathbb{F}_p$ is a perfect $\mathbb{F}_p$-algebra, it is not perfectoid at all since it is not Tate.

Proof. To see that 2 implies 1, note that A is a complete Tate ring by assumption, and that it is uniform[6]. The condition $\varpi^p \mid p$ is set to be vacuously true in characteristic p, so it suffices to check that
$$\mathrm{Frob}\colon A^\circ \to A^\circ/\varpi^p$$

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[6] More generally, and Tate ring in characteristic p which is complete is automatically uniform

is an isomorphism. By Lemma 5.5 we see that this is equivalent to requiring that Frobenius

$$\mathrm{Frob}\colon A^\circ \to A^\circ/p$$

to be an isomorphism, which is true by the assumption that A was perfect. to prove the converse, just apply Lemma 5.5 again. $\square$

A special class of perfectoid rings are those that happen to be fields, called perfectoid fields[7]. These admit a more concrete description, as in the theorem below.

Theorem 5.8 (Kedlaya)**.** *Let K be a complete topological field, then the followings are equivalent.*

1. *K is a perfectoid Tate ring.*
2. *The topology on K is induced by a rank one valuation*

$$|\cdot|\colon K \to \mathbb{R}_{\geq 0}$$

   such that

   - *The image of the valuation is non-discrete, i.e.* $\mathrm{im}(|\cdot|) \cap \mathbb{R}_{>0}$ *is not discrete*[8]*.*
   - $|p| < 1$.
   - *If $K^{\leq 1}$ denotes the subring of elements with valuation ≤ 1, then $K^{\leq 1}/p$ is semiperfect, i.e. has surjective Frobenius.*

Example 5.9.

- $\mathbb{Q}_p$ is not perfectoid, indeed its topology is induced by the usual p-adic valuation which has image $\{0\} \cup p^{\mathbb{Z}}$ (with the convention that $|p| = p^{-1}$). This is discrete as a subset of $\mathbb{R}_{>0}$. Note that $\mathbb{Q}_p^{\leq 1} \cong \mathbb{Z}_p$ is however semiperfect after reduction modulo p. Alternatively, note that one can not choose a pseudo-uniformiser ϖ of $\mathbb{Q}_p$ such that $\varpi^p \mid p$, since there are not enough p-th roots of unity.
- Define the Tate ring

$$\mathbb{Q}_p[p^{1/p^\infty}] = (\bigcup_n \mathbb{Q}_p[p^{1/p^n}])^\wedge,$$

  where each of the rings in the union is equipped with the valuation inherited from $\mathbb{Q}_p$, and the completion on the outside is taken with respect to this valuation. Note that $|p| = p^{-1}$ in any of these valuations. However, we see that the image of the valuation is not discrete, since the sequence

$$|p^{1/p^k}| = p^{-p^{-k}} \xrightarrow{k\to\infty} 1$$

  converges to the accumulation point 1 in the image. It suffices to check that $\mathbb{Q}_p[p^{1/p^\infty}]^{\leq 1}/p \cong \mathbb{F}_p$, which is clearly (semi-)perfect.
- Define the Tate ring

$$\mathbb{Q}_p^{\mathrm{cycl}} = (\bigcup_n \mathbb{Q}_p[\mu_{p^n}])^\wedge$$

  with the valuation once again inherited from $\mathbb{Q}_p$ under finite extensions and filtered colimits. This is once again perfectoid.

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[7] This terminology is not entirely universal

[8] We want to exclude zero, since this will often be a trivial accumulation point of the image in cases that are not perfectoid.

5.3 Tilting and Witt vectors

To distnguish between the notion of tilting for p-adically complete rings and perfectoid rings, we will introduce some nonstandard terminology.

Definition 5.10. Let R be a p-adically complete ring, then define the (integral) tilt of R to be the ring

$$R^\flat = \varprojlim_{\phi} R/p,$$

with the limit being taken along Frobenius maps.

It is clear that $R^\flat$, also known as the inverse limit perfection, is a perfect $\mathbb{F}_p$-algebra. Indeed, per construction the map

$$R^\flat \to R^\flat, (a_0, a_1, \dots) \mapsto (a_1, a_2, \dots)$$

is inverse to the Frobenius map on $R^\flat$.

Remark 5.11. The integral tilting construction above in fact induces an adjunction

$$\{\text{perfect } \mathbb{F}_p\text{-algebras}\} \underset{(-)^\flat}{\overset{W(-)}{\rightleftarrows}} \{p\text{-complete } \mathbb{Z}_p\text{-algebras}\},$$

where $W(-)$ denotes the p-typical Witt vectors. Furthermore, this adjunction is such that the unit

$$S \to W(S)^\flat$$

is always an equivalence. Given a p-complete $\mathbb{Z}_p$-algebra R, the counit map

$$\theta \colon W(R^\flat) \to R$$

is Fontaine's map.

Example 5.12. We have the canonical example

$$W(\mathbb{F}_p) = \mathbb{Z}_p,$$

and

Can't find a reference for this statement about A_{inf}

On the other hand, we can also define a notion of tilting for perfectoid rings.

Definition 5.13. Let A be a perfectoid ring, then define the (perfectoid) tilt of A to be the ring whose multiplicative monoid is given by

$$A^\flat = \varprojlim_{\phi} A,$$

the limit along the Frobenius, and addition defined by

$$(a^{(n)})_n + (b^{(n)})_n = (\lim_{k\to\infty} (a^{(n+k)} + b^{(n+k)})^{p^k})_n.$$

Lemma 5.14. *If A is perfectoid, then $A^\flat$ is perfectoid of characteristic p. If A itself was already of characteristic p, then $A \cong A^\flat$ is isomorphic to its tilt.*

Example 5.15. There is a chain of isomorphisms

$$(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \cong (\mathbb{F}_p(\!(t)\!)[t^{1/p^\infty}])^\wedge \cong (\mathbb{Q}_p[p^{1/p^\infty}])^\flat,$$

even though the two outside rings are not isomorphic before tilting.

Let us now extend the tilting construction to affinoid objects in perfectoid geometry.

Definition 5.16. A perfectoid Huber pair is a Huber pair (A, A^+) such that

1. A is perfectoid (and in particular Tate),
2. $A^+ \subset A^\circ$ is integral and open.

Define the tilt of a perfectoid Huber pair above as the perfectoid Huber pair $(A^\flat, A^{\flat +} = \varprojlim_{\phi} A^+)$.

Remark 5.17. In fact, we see that $A^{\flat +}$ is isomorphic to the integral tilt of A^+/ϖ.

Remark 5.18. If (A, A^+) is a perfectoid Huber pair, Fontaine's map

$$\theta\colon W(A^{\flat +}) \to A^+$$

is surjective, with kernel primitive of degree one, i.e. generated by an element of the form

$$p + [\varpi]\alpha$$

for $[\varpi]$ the multiplicative lift of ϖ to the Witt vectors, and α some element in $W(A^{\flat +})$. In fact, this assembles to an equivalence of categories

$$\begin{aligned} \{\text{perfectoid Huber pairs}\} &\cong \{\text{perfect prisms over } \mathbb{F}_p\} \\ (A, A^+) &\mapsto (A^\flat, A^{\flat +}, \ker(\theta)), \end{aligned}$$

where the right hand side is the category of triples consisting of a perfectoid Huber pair (R, R^+) in characteristic p and a primitive ideal $I \subset W(R^+)$ of degree one.

Let us remark that if (R, R^+, I) is a perfect prism over $\mathbb{F}_p$ and (S, S^+) is a perfectoid Huber pair with a map $f\colon (R, R^+) \to (S, S^+)$, then the image $f(I)$ is still primitive of degree one in $W(S^{\flat +})$. Let us now extract the main immediate consequence of this equivalence between perfectoid Huber pairs and perfect prisms.

Corollary 5.19 (Tilting equivalence)**.** *Let (A, A^+) be a perfectoid Huber pairs, then there are equivalences of categories (pHp = perfectoid Huber pair)*

$$\textit{pHp over } (A, A^+) \simeq \textit{perfect prisms over } (A^\flat, A^{\flat +}, \ker(\theta)) \simeq \textit{pHp over } (A^\flat, A^{\flat +}).$$

Proof. Tilting a perfectoid Huber pair that is already in characteristic p gives back the same result, so we just apply the tilting equivalence between perfectoid Huber pairs and perfect prisms twice. □

5.4 Perfectoid spaces

Now that we have discussed perfectoid Huber pairs, our affinoid perfectoids, let us globalise the theory to perfectoid spaces. First, note that if (A, A^+) is a perfectoid Huber pair, then it is automatically sheafy so that $\mathrm{Spa}(A, A^+)$ forms an affinoid adic space.

Definition 5.20. A perfectoid space is an adic space locally of the form $\mathrm{Spa}(A, A^+)$ for (A, A^+) a perfectoid Huber pair.

By applying the tilting equivalence on every perfectoid Huber pair and gluing this back together, we see that the tilting equivalence globalises to an equivalence

$$\mathrm{Pfd}_{/X} \simeq \mathrm{Pfd}_{/X^\flat}$$

of perfectoid spaces over a base perfectoid space X and its tilt $X^\flat$. In fact, we can strengthen this to the almost purity result of Faltings.

Theorem 5.21 (Almost purity)**.** *Let X be a perfectoid space, then there is an equivalence of sites*

$$X_{\text{ét}} \simeq X^\flat_{\text{ét}}.$$

Remark 5.22. Taking fundamental groups then recovers the Fontaine–Wintenberger theorem 5.1.

5.5 Pro-étale cohomology

Theorem 5.23. *Let (R, R^+) be a perfectoid Huber pair, and denote by $X = \mathrm{Spa}(R, R^+)$ the associated affinoid perfectoid adic space. Then*

- $\mathrm{H}^i(X; \mathcal{O}_X) = 0$ *for* $i > 0$,
- $\mathrm{H}^i(X; \mathcal{O}^+_X)$ *is almost zero for* $i > 0$, *and*
- $\mathrm{H}^i_{\text{proét}}(X; \widehat{\mathcal{O}}^+_X)$ *is almost zero for* $i > 0$.

Recall that an R^+-module M is almost zero, if for *any* pseudo-uniformiser[9] $\varpi \in R^+$, $\varpi M = 0$. We can use this to prove a surprising theorem about the interplay between the cohomology of adic spaces and affinoid perfectoids.

Theorem 5.24. *Let X be a locally Noetherian adic space over* $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. *Then the class of $U \in X_{\text{proét}}$ such that U is affinoid perfectoid forms a basis for the proétale topology on $X_{\text{proét}}$.*

Remark 5.25. A slightly stronger version of this, combined with Lemma 4.14 gives rise to the comparison between condensed and proétale cohomology in the previous talk.

5.6 How many untilts?

As we saw before, in the case of the Fontaine–Wintenberger isomorphism, nonisomorphic fields can have isomorphic tilts. It is therefore reasonable to ask for a moduli problem of untilts of a given perfectoid field of characteristic p.

Theorem 5.26 (Fargues–Fontaine)**.** *Let L be a perfectoid field of characteristic p. Then there exists a curve over $\mathbb{Q}_p$ (i.e. a regular Noetherian scheme of Krull dimension one) whose closed points are in bijection with (Frobenius orbits of isomorphism classes of) untilts of L, i.e. pair (K, ι) of a perfectoid field K of characteristic zero and a finite extension $\iota\colon L \to K^\flat$. The degree of a closed points is the degree of this extension.*

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[9] Note the word *any*, this helps us e.g. recover the more classical notions of being almost zero over a local ring in terms of all powers of the maximal ideal.

Note that we are taking Frobenius orbits, indeed there is a natural $\mathbb{Z}$-action on an untilt (K, ι), by sending ι to $\iota \circ \mathrm{Frob}^n$.

Proposition 5.27. *Given a perfectoid field L of characteristic p as above, there are infinitely many points of degree one on the Fargues–Fontaine curve over L.*

6 p-divisible groups (Vignesh Subramanian, 15 July)

7 Modifications of vector bundles (Kaif Hilman, 22 July)

7.1 Overview

We summarise first the story of the first three main theorems of [SW13].

Theorem 7.1 ([SW13, Thm. A])**.** *Let R be an f–semiperfect ring. Then the covariant Dieudonne module functor on p–divisible groups over R up to isogeny is fully faithful.*

Of particular interest are morphisms $\mathbb{Q}_p/\mathbb{Z}_p \to G$. So we study $\widetilde{G} := \mathrm{Hom}_S(\mathbb{Q}_p/\mathbb{Z}_p, G)[p^{-1}]$ called the universal cover assembling to a functor from R–algebras to $\mathbb{Q}_p$–vector spaces. An important property of this construction is its crystalline nature, i.e. if $S \to R$ is a surjection with nilpotent kernel, and if G_S is a lift of G to S, then $\widetilde{G}_S(S) \cong \widetilde{G}(R)$ canonically (and so does not depend on the choice of lift G_S). That is, $\widetilde{G}$ is a crystal on the infinitesimal site of R. Via this construction, we will see how Theorem 7.1 is closely related to p–adic Hodge theory.

Take $R = \mathcal{O}_C/p$ where C is an algebraically closed complete extension of $\mathbb{Q}_p$ and let G be any p–divisible group over $\mathcal{O}_C$ with reduction G_0 to $\mathcal{O}_C/p$. Let $M(G)$ be the finite projective B^+_{cris}–module given by evaluating the Diedonne crystal of G on $A_{\text{cris}} = A_{\text{cris}}(\mathcal{O}_C/p)$ and inverting p. Via Theorem 7.1, one finds that

$$M(G)^{\varphi=p} = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G_0)[p^{-1}] = \widetilde{G}(\mathcal{O}_C/p) = \widetilde{G}(\mathcal{O}_C)$$

which recovers the rational isomorphism in p–adic Hodge theory for p–divisible groups. Thus, via the exact sequence

$$0 \longrightarrow T(G)[p^{-1}] \longrightarrow \widetilde{G}(\mathcal{O}_C) \longrightarrow \mathrm{Lie}\, G \otimes C \longrightarrow \tag{7.2}$$

with the universal cover in the middle and the log map on the right, the identification above gives us the exact sequence

$$0 \longrightarrow T(G)[p^{-1}] \longrightarrow M(G)^{\varphi=p} \longrightarrow \mathrm{Lie}\, G \otimes C \longrightarrow 0$$

relating the etale (which is the kernel) and the crystalline homology (the middle term) of G. Note that while the middle term does not depend on the lift G of G_0, the outer terms do.

The next main result, which will be the main focus of this talk, is the linear algebraic classification of p–divisible groups. Here, as before, C will be an algebraically closed complete extension of $\mathbb{Q}_p$. We recall that there is a Hodge–Tate sequence

$$0 \longrightarrow \mathrm{Lie}\, G \otimes C(1) \longrightarrow T(G) \otimes C \longrightarrow (\mathrm{Lie}\, G^\vee)^\vee \otimes C \longrightarrow 0$$

where (1) is the Tate twist.

Theorem 7.3 ([SW13, Thm. B])**.** *There is an equivalence of categories between the category of p–divisible groups over $\mathcal{O}_C$ and the category of free $\mathbb{Z}_p$–modules T of finite rank together with a C–subvector space W of $T \otimes C(-1)$ given by sending G to $(T, W) = (TG, \operatorname{Lie} G \otimes C)$.*

A salient point is that this is a classification in terms of linear algebra instead of σ–linear algebra as is the usual case, and (related to that) is in terms of its etale cohomology instead of its crystalline cohomology.

The functor stated in the theorem will be proved to be fully faithful by hand, and then again by hand, proved to be essentially surjective in the case when C is spherically complete with surjective norm $C \to \mathbb{R}_{\geq 0}$. To prove that it is essentially surjective in general, it will be crucial to relate (7.2) to the Fargues–Fontaine curve.

There is a special point $\infty \in X_{\mathrm{FF}}$ corresponding to C. For every isocrystal M over $\overline{\mathbb{F}}_p$, there is a corresponding vector bundle on X_{FF}, given by the vector bundle associated to $\bigoplus_{d\geq 0}(M \otimes B^+_{\mathrm{cris}})^{\varphi = p^d}$ and Fargues–Fontaine show that basically all vector bundles arise in this way. Combining Fargues–Fontaine's classification and Theorem 7.1, it is shown in their Theorem 5.1.4 that every p–divisible group over $\mathcal{O}_C$ is *isotrivial* in that there exists a p–divisible group H over $\overline{\mathbb{F}}_p$ and a quasi–isogeny

$$\rho\colon H \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p \longrightarrow G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p.$$

Thus (G, ρ) is a deformation of H in the sense of Rapoport–Zink. By this combination of results, it is shown also that there is a bijection between isogeny classes of p–divisible gropus over $\overline{\mathbb{F}}_p$ and vector bundles on X whose slopes are between 0 and 1. Note importantly also, by construction, that $i_\infty^* \mathcal{E}(H) = M(H) \otimes C$.

Again by Theorem 7.1, we can identify $\widetilde{G}(\mathcal{O}_C) \cong \widetilde{H}(\mathcal{O}_C/p) \cong (M(H) \otimes B^+_{\mathrm{cris}})^{\varphi = p}$ with the global sections of $\mathcal{E}(H)$. Thus, the exact sequence (7.2) is identifies with the global sections of the modification of vector bundles

$$0 \longrightarrow TG \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}} \longrightarrow \mathcal{E}(H) \longrightarrow i_{\infty *}(\operatorname{Lie} G \otimes C) \longrightarrow 0$$

All of this is explained in Proposition 7.36. To summarise the situation, a p–divisible group H over $\overline{F}_p$ gives rise to a vector bundle on X_{FF}, whereas a deformation of it to $\mathcal{O}_C$ gives rise to a *(minuscule) modification* of $\mathcal{E}$.

Finally, we explain the third character in the talk, namely the Grothendieck–Messing period morphism, which will allow us bootstrap via the Fargues–Fontaine curve the special case of Theorem 7.3 to prove the general case.

Let H be a p–divisible group of dimension d and height h over a perfect field k of characteristic p. Let $\mathcal{M}$ be the associated Rapoport–Zink space. This is a formal scheme parametrising deformations (G, ρ) of H, i.e. it looks almost like isotriviality above but works only for nilpotent $W(k)$–algebras R instead of $\mathcal{O}_C$. Passing to generic fibres we obtain an adic space $\mathcal{M}^{\mathrm{ad}}_\eta$. It is important not to consider it just as a rigid space but actually as adic space since it is important for us to consider nonclassical points.

There is then a Grothendieck–Messing period map $\pi\colon \mathcal{M}^{\mathrm{ad}}_\eta \to \mathcal{F}\ell$ of adic spaces, where $\mathcal{F}\ell$ is the Grassmannian variety of d–dimensional quotients of $M(H) \otimes \mathbb{Q}_p$. If x is a $(C, \mathcal{O}_C)$–point of $\mathcal{M}^{\mathrm{ad}}_\eta$ corresponding to a deformation (G, ρ) of H to $\mathcal{O}_C$, then $\pi x \in \mathcal{F}\ell(C, \mathcal{O}_C)$ corresponds to the quotient $M(H) \otimes C \to W$ given by Grothendieck–Messing theory.

The third main theorem is a criterion for being in the image of the period map. If x is a $(C, \mathcal{O}_C)$–point of $\mathcal{F}\ell$ corresponding to a quotient $M(H) \otimes C \to W$, we may construct a surjection of sheaves

$$\mathcal{E}(H) \longrightarrow i_{\infty *} i_\infty^* \mathcal{E}(H) \simeq i_{\infty *}(M(H) \otimes C) \longrightarrow i_{\infty *} W \longrightarrow 0.$$

Let $\mathcal{F}$ be the kernel.

Theorem 7.4 ([SW13, Thm. C]). *The point x is in the image of π if and only if $\mathcal{F}$ is trivial, i.e. $\mathcal{F} \cong \mathcal{O}_{X_{\mathrm{FF}}}^{\oplus h}$.*

To prove this theorem, we may reduce to the case when C is spherically complete with surjective norm map. In that case, Theorem 7.3 in this special case of C constructs the desired p–divisible group G. The construction of the quasi–isogeny is then supplied by their Theorem 5.1.4. We may then use Theorem 7.4 to deduce the general case of essential surjectivity of Theorem 7.3.

7.2 Complements on p-divisible groups

7.2.1 Basic notions and constructions

This is based on [Hon20, §II.3.2, II.3.3]. Fix a base ring $R = \mathcal{O}_K$ and let L be the p–adic completion of an algebraic extension of K and denote by $\mathfrak{m}_L$ its maximal ideal. We are particularly interested in the case where $L = \mathbb{C}_K$.

Definition 7.5. Let $G = \operatorname{colim}_n G_n$ be a p–divisible group over $\mathcal{O}_K$. We define

$$G(\mathcal{O}_L) := \lim_i G(\mathcal{O}_L/\mathfrak{m}^i) = \lim_i \operatorname*{colim}_n G_n(\mathcal{O}_L/\mathfrak{m}^i).$$

Corollary 7.6 ([Hon20, Cor. 3.2.4]). *Let G be a connected p–divisible group of dimension d over $\mathcal{O}_K$. We have a canonical isomorphism of $\mathbb{Z}_p$–modules*

$$G(\mathcal{O}_L) \cong \operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{O}_K[[t_1, \dots, t_d]], \mathcal{O}_L)$$

where the multiplication by p on the right hand side is induced by $[p]_{\mu(G)}$.

Definition 7.7. Let G be a p–divisible group over $\mathcal{O}_K$ of dimension d, $\mathcal{A}^\circ \cong \mathcal{O}_K[[t_1, \dots, t_d]]$ be the formal group associated to the identity component of G and I the augmentation ideal.

(1) Given an $\mathcal{O}_K$–module M, we define the *tangent space* and *cotangent space* of G with M–coefficient by

$$t_G M := \operatorname{Hom}_{\mathcal{O}_K}(I/I^2, M) \qquad t_G^* M := I/I^2 \otimes_{\mathcal{O}_K} M.$$

(2) For all real numbers $\lambda > 0$, we define the *valuation filtration of* $G^\circ(\mathcal{O}_L)$ by

$$\operatorname{Fil}^\lambda G^\circ(\mathcal{O}_L) := \{f \in G^\circ(\mathcal{O}_L) \ : \ \nu(f(x)) \geq \lambda \text{ for all } x \in I\}$$

where we have used the identification $G^\circ(\mathcal{O}_L) \cong \operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{A}^\circ, \mathcal{O}_L)$ as in Corollary 7.6.

Definition 7.8. The Lie algebra of $G^\circ(\mathcal{O}_L)$ is defined to be $\operatorname{Lie} G(\mathcal{O}_L) := t_G(L)$.

Lemma 7.9 ([Hon20, Lem. 3.3.3]). *Let G be a p–divisible gropu over $\mathcal{O}_K$ and denote I for its augmentation ideal. For any $f \in G(\mathcal{O}_L)$ and $x \in I$, the limit $\lim_{n\to\infty} \frac{(p^n f)(x)}{p^n}$ exists in L and equals zero if $x \in I^2$.*

Definition 7.10. Let M be a Gal_K–module. We define the Tate twists $M(n)$ for $n \in \mathbb{Z}$ as

$$M(n) := \begin{cases} M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes n} & \text{if } n \geq 0 \\ \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(1)^{\otimes -n}, M) & \text{if } n < 0. \end{cases}$$

Proposition 7.11 ([Hon20, Lem. II.3.1.10])**.** *Let M be a $\mathbb{Z}_p[\mathrm{Gal}_K]$–module. For each $m, n \in \mathbb{Z}$, we have natural Gal_K–isomorphisms*

$$M(m) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n) \cong M(m+n) \qquad M(n)^\vee \cong M^\vee(-n).$$

Definition 7.12 ([Sha19, pg. 9])**.** For any two p–divisible groups G, H, we write

$$q\mathrm{Hom}(G,H) := \mathrm{Hom}(G,H)[p^{-1}].$$

The *category of p–divisible groups up to isogeny* is defined to have the same objects as the category of p–divisible groups but with Hom replaced by qHom. A *quasi–isogeny* is an isomorphism in this category, and an *isogeny* is a quasi–isogeny which is a homomorphism.

Remark 7.13. If G and H are isogenous, then they have the same height and dimension.

Remark 7.14. By definition of qHom, we see that any morphism f in the category of p–divisible groups up to isogeny satisfies that $p^n f$ is an actual homomorphism between p–divisible groups.

Construction 7.15 (Universal vectorial extension, [Ban+22, de Shalit §5.3.5])**.** Let G be a p–divisible group over R and suppose first that $p^N = 0 \in R$. In particular, we know that $p^N \mathbb{G}_a = 0$. Now for any $n \geq N$, we may compute

$$\mathrm{Ext}(G, \mathbb{G}_a) \cong \mathrm{Ext}(G, \mathbb{G}_a)[p^n] \cong \mathrm{Hom}(G[p^n], \mathbb{G}_a) = \{a \in A_n \mid m_G^*(a) = a \otimes 1 + 1 \otimes a\}.$$

But now, viewing $A_n = \mathrm{Hom}(A_n^\vee, R)$, the primitive elements therein when regarded as linear functionals $A_n^\vee \to R$ define ring homomorphisms

$$A_n^\vee \longrightarrow R[\epsilon]/(\epsilon^2)$$

which reduces module ϵ to the counit homomorphism (the dual of the structure map $R \to A_n$). This is the same as defining an element in $\mathrm{Lie}(G^\vee[p^n]) \cong \ker\big(G^\vee[p^n](R[\epsilon]/(\epsilon^2)) \to G^\vee[p^n](R)\big)$. Thus, we obtain further identifications

$$\mathrm{Ext}(G, \mathbb{G}_a) \cong \{a \in A_n \mid m_G^*(a) = a \otimes 1 + 1 \otimes a\} \cong \mathrm{Lie}(G^\vee[p^n]) \cong \mathrm{Lie}(G^\vee) \cong \mathrm{Hom}(\omega_{G^\vee}, \mathbb{G}_a).$$

Hence, for any finitely generated free R–module M, we obtain an isomorphism

$$\mathrm{Ext}(G, \underline{M}) \cong \mathrm{Hom}(\omega_{G^\vee}, M).$$

In particular, setting $M = \omega_{G^\vee}$, we obtain a *universal vectorial extension*

$$0 \longrightarrow \omega_{G^\vee} \longrightarrow EG \longrightarrow G \longrightarrow 0$$

coming from the identity map on $\omega_{G^\vee}$. We then also define

$$MG := \mathrm{Lie}\, EG.$$

7.2.2 Key yoga of p-divisible groups

In this subsubsection, we collect together the four maps associated to p–divisible groups that will play a fundamental role in this talk. For a p–divisible group G, these are the maps of group schemes

$$\begin{array}{ccl} T_p(G^\vee) & \xrightarrow{\cong} & \operatorname{Hom}_{\mathbb{Z}_p}(T_pG, \mathbb{Z}_p(1)) = (T_pG)^\vee(1) \\ G & \xrightarrow{\log_G} & \operatorname{Lie} G \\ T_pG & \xrightarrow{\alpha_G} & \big(\operatorname{Lie}(G^\vee)\big)^\vee = \operatorname{Hom}(\operatorname{Lie}(G^\vee), \mathbb{G}_a) \\ MG & \xrightarrow{q_G} & \operatorname{Lie} G \end{array}$$

called the *Cartier duality for Tate modules,* the *logarithm*, the *Hodge–Tate period map,* and the *Hodge–de Rham (or Grothendieck–Messing) period map*, respectively. The last two maps are related, and this connection underpins the duality that we will see in the next talk.

We first deal with the duality on Tate modules.

Construction 7.16 (Cartier duality on Tate modules, [Sti09, §11.2.2])**.** Let $G = (G_n)_n$ be a p–divisible group. Recall that by Cartier duality on finite flat group schemes, we have a natural isomorphism $G_n^\vee \cong \operatorname{Hom}(G_n, \mu_{p^n})$ induced by a perfect pairing

$$G_n^\vee \times G_n \xrightarrow{\langle -,- \rangle} \mu_{p^n}.$$

It turns out that these pairings are compatible in that we have

$$\begin{tikzcd} G_{t+n} \arrow[d, "j_{t,n}"'] & \times & G_{t+n}^\vee \arrow[r] & \mu_{p^{r+n}} \\ G_n & \times & G_n^\vee \arrow[u, "i_{t,n}"'] \arrow[r] & \mu_{p^n} \arrow[u, "i_{t,n}"] \end{tikzcd}$$

because $i_{G^\vee,t,n} = j_{G,t,n}^\vee$. Thus, for $(x_n) \in T_pG$ and $(y_n) \in T_p(G^\vee)$, we see that

$$\langle x_n, y_n \rangle = \langle j_{t,n}(x_{t+n}), \langle j_{t,n}(y_{t+n}) \rangle = \langle x_{t+n}, i_{t,n} j_{t,n}(y_{t,n}) \rangle = \langle x_{t+n}, p^t y_{t+n} \rangle = \langle x_{t+n}, y_{t+n} \rangle^{p^t}$$

in $\mu_{p^n} \subseteq \mu_{p^{t+n}}$. Thus, we may obtain a well–defined pairing

$$T_pG \times T_p(G^\vee) \xrightarrow{\langle -,- \rangle} T_p\mu_{p^\infty} = \mathbb{Z}_p(1) \quad :: \quad ((x_n),(y_n)) \mapsto (\langle x_n, y_n \rangle)$$

which is levelwise a perfect pairing, and hence a perfect pairing itself. Furthermore, all maps constructed so far are clearly Gal_K–equivariant, and so adjointing over thus yields the isomorphism

$$T_p(G^\vee) \xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}_p}(T_pG, \mathbb{Z}_p(1)) = (T_pG)^\vee(1)$$

of $\mathbb{Z}_p[\operatorname{Gal}_K]$–modules, as desired.

Next, we recount the construction of the logarithm and summarise its key properties.

Construction 7.17 (Logarithms)**.** Let G be a p–divisible group over $\mathcal{O}_K$ and let I denote the augmentation ideal. We define the logarithm of G to be the map

$$\log_G \colon G(\mathcal{O}_L) \longrightarrow t_G L$$

defined as, for every $f \in G(\mathcal{O}_L) \cong \operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{A}^\circ, \mathcal{O}_L)$ and $x \in I/I^2$, writing $\widetilde{x} \in I$ for a lift of x,

$$\log_G(f)(x) := \lim_{n \to \infty} \frac{(p^n f)(\widetilde{x})}{p^n}.$$

Here are some fundamental results about the logarithm which will be helpful to fix our intuition about it.

Proposition 7.18 ([Hon20, Prop. 3.3.6]). *Let G be a p–divisible group over $\mathcal{O}_K$. Denote by I the augmentation ideal.*

(1) $\log_G$ *is a group homomorphism,*

(2) $\log_G$ *is a local isomorphism in the sense that for each real number $\lambda \geq 1$, it induces an isomorphism*

$$\mathrm{Fil}^{\lambda} G^{\circ}(\mathcal{O}_L) \xrightarrow{\simeq} \left\{ \tau \in t_G L \ : \ \nu(\tau(x)) \geq \lambda \textit{ for allx} \in I/I^2 \right\}.$$

(3) We have an identification $\ker(\log_G) = G(\mathcal{O}_L)_{\mathrm{tors}} \subset G(\mathcal{O}_L)$.

(4) $\log_G$ *induces an isomorphism $G(\mathcal{O}_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong t_G L$.*

We now give the constructions of the two period maps.

Construction 7.19 (Hodge–Tate period map)**.** Recall first that we have $(\underline{\mathbb{Q}_p/\mathbb{Z}_p})^{\vee} \cong \mu_{p^\infty}$ and that $\mathrm{Lie}\, \mu_{p^\infty} \cong \mathbb{G}_a$. We define the Hodge–Tate map

$$\alpha_G \colon T_p G \longrightarrow (\mathrm{Lie}(G^{\vee}))^{\vee}$$

as the following composite

$$T_p G \coloneqq \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{(-)^{\vee}} \mathrm{Hom}(G^{\vee}, \mu_{p^\infty}) \xrightarrow{\mathrm{Lie}} \mathrm{Hom}(\mathrm{Lie}(G^{\vee}), \mathbb{G}_a) = (\mathrm{Lie}(G^{\vee}))^{\vee}.$$

An important associated manoevure is to dualise this, namely, consider the C–basechanged $\alpha_{G^{\vee}}$ map

$$\alpha_{G^{\vee}} \colon T_p(G^{\vee}) \otimes_{\mathbb{Z}_p} C \longrightarrow (\mathrm{Lie}\, G)^{\vee} = \mathrm{Hom}(\mathrm{Lie}\, G, \mathbb{G}_a).$$

Now using that $T_p(G^{\vee}) \otimes C \cong T_p G(-1) \otimes C \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p G, \mathbb{Z}_p(1)) \otimes C \cong \mathrm{Hom}_C(T_p G \otimes C, C(1))$, we may dualise $\alpha_{G^{\vee}}$ by applying $\mathrm{Hom}_C(-, C)$ to obtain

$$\alpha_{G^{\vee}}^{\vee} \colon \ \mathrm{Lie}\, G \longrightarrow T_p G \otimes C(-1).$$

The celebrated Hodge–Tate sequence then says that we have a short exact sequence

$$0 \longrightarrow \mathrm{Lie}\, G \otimes C(1) \xrightarrow{\alpha_{G^{\vee}}^{\vee}(1)} T \otimes C \xrightarrow{\alpha_C} (\mathrm{Lie}\, G^{\vee} \otimes C)^{\vee} \longrightarrow 0.$$

Construction 7.20 (Hodge–de Rham period map)**.** Recall that we had the universal vector extension

$$0 \longrightarrow \underline{\omega}_{G^{\vee}} \longrightarrow EG \xrightarrow{\mathrm{can}} G \longrightarrow 0.$$

Applying the Lie algebra functor yields a short exact sequence

$$0 \longrightarrow \underline{\omega}_{G^{\vee}} \longrightarrow MG \xrightarrow{q_G} \mathrm{Lie}\, G \longrightarrow 0.$$

The *Hodge–de Rham period map* q_G is then defined as the right hand surjection.

7.3 The curve

7.3.1 Overview

This overview is more or less based on the very accessible and enlightening notes [Mor18] by Morrow. Another similarly nice introduction to this is de Shalit's [Ban+22, §1]. The original material (pardon their French!) is of course [FF18].

Recall that a perfectoid field is roughly a nonarchimedean field C with nondiscrete valuation, $|p| < 1$ (so that $p \in \mathfrak{m}_{\mathcal{O}_C}$ where $\mathcal{O}_C = C^{\leq 1}$), and where the Frobenius $(-)^p \colon \mathcal{O}_C/p \to \mathcal{O}_C/p$ is surjective. Recall also that we had the tilt construction

$$C^\flat := \lim \big(\cdots \xrightarrow{(-)^p} C \xrightarrow{(-)^p} C \big)$$

where it can be shown that, if C were perfectoid, then $C^\flat$ carries a natural ring structure and is a perfectoid field of characteristic p. One can then ask the following:

Question: *For a fixed algebraically closed perfectoid field F of characteristic p, how many perfectoid fields are there that tilt to F? In other words, what is the "space of untilts" of F?*

More precisely, write $|Y_F|$ for the set of untilts (C, ι) of F, i.e. such tuples where C is a perfectoid field and $\iota F \xrightarrow{\cong} C^\flat$ is a specified ring isomorphism. This set has an action of the Frobenius since we may twist the isomorphism ι with the Frobenius $\varphi \colon F \xrightarrow{\cong} F$. We may thus wonder about the "space" $|Y_F|/\varphi^{\mathbb{Z}}$.It turns out that there is a unique untilt of characteristic p, and so the set $|Y_F|/\varphi^{\mathbb{Z}}$ has a distinguished point. Now, the surprising insight of Fargues and Fontaine is that, while this looks like a hopelessly complicated problem, it is scheme–theoretically extremely simple (relative to its importance) since it is just a curve!

Theorem 7.21 ([FF18, Thm. 6.5.2])**.** *The set $|Y_F|/\varphi^{\mathbb{Z}}$ is the underlying set of points of a curve X_{FF}, now called the Fargues–Fontaine curve.*

This curve is now revered as the fundamental curve of p–adic Hodge theory, and its construction may be viewed as a geometrisation of Fontaine's famous zoo of so–called "period rings". It shares many similarities with the very basic curve $\mathbb{P}^1_{\mathbb{C}}$, and for the remainder of the overview, we shall remind the reader of the salient features of the curve $\mathbb{P}^1_{\mathbb{C}}$ in order to motivate the rather convoluted construction of X_{FF} in terms of Fontaine's period rings.

Recollection 7.22 ([Ban+22, de Shalit, §1.1.2], [Mor18, §2.3])**.** We recall the following standard facts about the curve $\mathbb{P}^1_{\mathbb{C}}$:

(1) It may be constructed as

$$\mathbb{P}^1_{\mathbb{C}} := \mathrm{Proj}\Big(\bigoplus_{k \geq 0} \mathrm{Fil}_k \mathbb{C}[z]\Big)$$

where $\mathrm{Fil}_k\mathbb{C}[z] := \{f \in \mathbb{C}[z] \ : \ \deg f \leq k\}$. In particular, this is a curve over $\mathrm{Spec}(\mathbb{C})$.

(2) More geometrically, it may also be viewed as gluing the line $\mathrm{Spec}(\mathbb{C}[z])$ and the formal disk at infinity $\mathrm{Spec}(\mathbb{C}[[z^{-1}]])$ along the punctured formal disk $\mathrm{Spec}(\mathbb{C}((z^{-1})))$.

(3) There is a fundamental exact sequence of vector spaces

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}[z] \longrightarrow \mathbb{C}((z^{-1}))/\mathbb{C}[[z^{-1}]] \longrightarrow 0.$$

(4) There are line bundles $\mathcal{O}(n)$ for all $n \in \mathbb{Z}$ on $\mathbb{P}^1_{\mathbb{C}}$ and every vector bundle is isomorphic to a unique direct sum $\bigoplus_{i=1}^k \mathcal{O}(n_i)$. These are defined by the shifts of the $\bigoplus_{k\geq 0} \mathrm{Fil}_k \mathbb{C}[z]$–graded module.

To describe the Fargues–Fontaine curve (or henceforth just "the curve"), we summarise the period rings that will be involved. These should be viewed as the appropriate replacements of the rings of functions $\mathbb{C}[z]$, $\mathbb{C}[[z^{-1}]]$, and $\mathbb{C}((z^{-1}))$. The following construction is a "shortcut" version of defining the period rings that go into constructing the curve, and we should note here that the way Fargues–Fontaine did it (which is the "standard" way to do it) proceeds a bit differently using slightly different period rings.

Construction 7.23 (Summary of period rings, [Mor18, §5.3.2])**.** Fix an algebraically closed perfectoid field F of characteristic p. The one ring to rule them all is $\mathbf{A}_{\mathrm{inf}} := W(\mathcal{O}_F)$ (note that F will always be implicit here since it is fixed once and for all), and all the other period rings will issue from this. One indication as to how this ring can have a chance at governing the question of untilts posed earlier is the result (c.f. for example [Mor18, Prop. 5.1] for a writeup) that there is a natural bijection between $|Y_F|$ and the set of ideals of $\mathbf{A}_{\mathrm{inf}}$ generated by a "primitive element of degree one". So fix an untilt C of characteristic 0 and the element $\xi \in \mathbf{A}_{\mathrm{inf}}$ corresponding to it under the result aforementioned. We will be viewing this point $C \in |Y_F|$ as a chosen point ∞ on X_{FF} later.

First, we define the crystalline period ring B^+_{cris} as the p–inverted p–completion of the PD thickening of $\mathbf{A}_{\mathrm{inf}}$ associated to ξ, i.e.

$$B^+_{\mathrm{cris}} := \mathbf{A}_{\mathrm{inf}}\widehat{\Big[\frac{\xi^n}{n!} : n \geq 1\Big]}\Big[\frac{1}{p}\Big].$$

It will turn out that there is a distinguished element $t = t_\infty \in B^{+,\varphi=p}_{\mathrm{cris}}$ that will "correspond" to the untilt C. We then define $B_{\mathrm{cris}} := B^+_{\mathrm{cris}}[\frac{1}{t}]$ and

$$B_e := B^{\varphi=1}_{\mathrm{cris}}.$$

Next, we define the de Rham period rings. Define B^+_{dR} to be the ξ–adic completion of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}]$. Via the canonical ring map $\mathbf{A}_{\mathrm{inf}}\widehat{[\frac{\xi^n}{n!} : n \geq 1]}[\frac{1}{p}] \to \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}]$, we may view t as an element in $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}]$, and hence also in B^+_{dR}. We then define

$$B_{\mathrm{dR}} := B^+_{\mathrm{dR}}\Big[\frac{1}{t}\Big].$$

Given these, we may then give the definition of the curve.

Notation 7.24. To make the notations less clunky, we will also write $P := \bigoplus_{k\geq 0} B^{+,\varphi=p^k}_{\mathrm{cris}}$ and $P_d = B^{+,\varphi=p^d}_{\mathrm{cris}}$. For $d \in \mathbb{Z}$, we will also define the P–graded module $P(d)$ as the appropriate d–shift of the graded module P.

Definition 7.25. The *Fargues–Fontaine curve* associated to the characteristic p algebraically closed perfectoid field F is defined as

$$X_{\mathrm{FF}} := X^F_{\mathrm{FF}} := \mathrm{Proj}\Big(\bigoplus_{k\geq 0} B^{+,\varphi=p^k}_{\mathrm{cris}}\Big) = \mathrm{Proj}(P) = \mathrm{Proj}\Big(\bigoplus_{k\geq 0} P_k\Big).$$

The fundamental theorem of Fargues–Fontaine is that this is a curve, but they did much more. Here is a summary of some basic results on the curve, emphasising the analogy with $\mathbb{P}^1_{\mathbb{C}}$ from Recollection 7.22.

Fact 7.26 (Some basic results on the curve)**.** Let F be an algebraically closed characteristic p perfectoid field, $X_{\mathrm{FF}} = X_{\mathrm{FF}}^F$ the associated Fargues–Fontaine curve, and C a characteristic 0 untilt of F. Recall also the notation $P = \bigoplus_{k\geq 0} P_k = \bigoplus_{k\geq 0} B_{\mathrm{cris}}^{+,\varphi=p^k}$.

(1) We have $P_0 \cong \mathbb{Q}_p$, and so X_{FF} is a curve over $\mathbb{Q}_p$.

(2) We may view X_{FF} as being obtained by gluing the line $\mathrm{Spec}(B_e)$ with the formal disk $\mathrm{Spec}(B_{\mathrm{dR}}^+)$ at ∞ along $\mathrm{Spec}(B_{\mathrm{dR}})$. As such, B_{dR}^+ should be viewed as the completed local ring at $\infty \in X_{\mathrm{FF}}$. Furthermore, the element $t \in P$ from Construction 7.23 is a nonzero divisor, and we have $P_{d+1}/tP_d \cong C$ (c.f. the proof of [Hon20, Prop. IV.3.2.8]). Moreover, by [Lur18, Prop. 8.11], we know that B_{dR}^+ is a discrete valuation ring with uniformiser ξ which is not a zero–divisor and $B_{\mathrm{dR}}^+/\xi \cong C$.

(3) Associated to this point $\infty \in X_{\mathrm{FF}}$ is the *fundamental exact sequence of p–adic Hodge theory at C*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B^e \longrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \longrightarrow 0.$$

(4) There are line bundles $\mathcal{O}_{X_{\mathrm{FF}}}(d)$ for all $d \in \mathbb{Z}$ on X_{FF} defined using the shifted P–graded modules $P(d)$ from Notation 7.24. Moreover, $H^1(X_{\mathrm{FF}}, \mathcal{O}_{X_{\mathrm{FF}}}) = 0$.

Unlike the case of $\mathbb{P}^1_{\mathbb{C}}$ though, the theory of vector bundles on the curve is quite a bit more complicated. One of the many crowning achievements of Fargues and Fontaine is that they completely classified the possible vector bundles on their curve, and this requires the theory of p–divisible groups and isocrystals. It turns out that there is a method to associate to an isocrystal (D, φ) a vector bundle $\mathcal{E}(D, \varphi)$, and this leads to their major classification result.

Theorem 7.27 (Vector bundles are isocrystals, [FF18, Thm. 8.2.10], [Mor18, Thm. 3.4], [Ban+22, Thm. 59])**.** *Let E be a vector bundle on X_{FF}. Then there exists a unique sequence of rational numbers $\lambda_1 \geq \cdots \geq \lambda_m$ such that E is isomorphic to $\bigoplus_{i=1}^m \mathcal{O}_{X_{\mathrm{FF}}}(\lambda_i)$. In other words, every vector bundle over X_{FF} is of the form $\mathcal{E}(D, \varphi_D)$ for a unique isocrystal (D, φ_D).*

We will explain in detail this procedure of obtaining vector bundles on X_{FF} from objects related to p–divisible groups in the next section via the so–called *(minuscule) modification* construction. It will play a central role in many of the proofs in the remainder of this note.

7.3.2 Vector bundle interpretation of p-divisible groups over the special fibre

As a consequence of the full faithfulness result over $\mathcal{O}_C/p$, we recover the rational comparison isomorphism for p–divisible groups.

Corollary 7.28 ([SW13, Cor. 5.1.2])**.** *Let G be a p–divisible group over $\mathcal{O}_C$ and let $T := T(G)(\mathcal{O}_C)$. The sequence*

$$0 \longrightarrow T[p^{-1}] \longrightarrow \widetilde{G}(\mathcal{O}_C) \longrightarrow \mathrm{Lie}\, G \otimes C \longrightarrow 0$$

is exact, and gets identified with the sequence

$$0 \longrightarrow T[p^{-1}] \longrightarrow (M[p^{-1}])^{\varphi=p} \longrightarrow \mathrm{Lie}\, G \otimes C \longrightarrow 0.$$

Proof. To see the identification, just note that

$$\widehat{G}(\mathcal{O}_C) \cong \widetilde{G}(\mathcal{O}_C/p) \cong \mathrm{Hom}_{\mathcal{O}_C/p}(\mathbb{Q}_p/\mathbb{Z}_p, G)[p^{-1}] \cong (M[p^{-1}])^{\varphi=p}$$

where the first isomorphism is by the crystalline nature of the universal cover and the second isomorphism by definition. To see right exactness, just note that C is algebraically closed, and so multiplication by p on $G(\mathcal{O}_C)$ is surjective. □

In fact, the full faithfulness Dieudonne theory we had implies the following:

Theorem 7.29 ([SW13, Thm. 5.1.4])**.**

(1) Let G be a p–divisible group over $\mathcal{O}_C$. Then there exists a p–divisible group H over $\overline{\mathbb{F}}_p$ and a quasi–isogeny

$$\rho\colon H \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p \longrightarrow G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p.$$

In other words, all p–divisible groups over $\mathcal{O}_C$ are isotrivial in the sense of Fargues.

(2) The functor $G_0 \mapsto \mathcal{E}(G_0)$ from p–divisible groups over $\mathcal{O}_C/p$ up to isogeny to vector bundles over X_{FF} is fully faithful. The essential image is given by the vector bundles all of whose slopes are between 0 and 1.

Sketch proof. Since $\mathcal{O}_C/p$ is f–semiperfect, [SW13, Thm. 4.1.4] reduces the result to the analogous statement for Dieudonne modules over B^+_{cris}. Then both results are due to Fargue–Fontaine. ☐

7.3.3 Vector bundles and modifications on the curve

The following is based on the discussions in [FF18, §8.3.2.1] and [Hon20, Proof of Prop. IV.3.2.8].

Construction 7.30 (The fundamental exact sequence of p–adic Hodge theory)**.** Since P is an integral domain, letting $t \in P_1$ be the element corresponding to C, we obtain a short exact sequence of graded P–modules

$$0 \longrightarrow P(d) \xrightarrow{t} P(d+1) \longrightarrow P(d+1)/tP(d) \longrightarrow 0.$$

This gives rise to an exact sequence of coherent $\mathcal{O}_{X_{\mathrm{FF}}}$–modules

$$0 \longrightarrow \mathcal{O}_{X_{\mathrm{FF}}}(d) \xrightarrow{t} \mathcal{O}_{X_{\mathrm{FF}}}(d+1) \longrightarrow \mathcal{O}_{X_{\mathrm{FF}}}(d+1)/t\mathcal{O}_{X_{\mathrm{FF}}}(d) \longrightarrow 0.$$

Since $P(d) \xrightarrow{\cdot t} P(d+1)$ is an isomorphism upon inverting t, we know that the coherent $\mathcal{O}_{X_{\mathrm{FF}}}$–module is a skyscraper sheaf supported at ∞. But then

$$H^0(X_{\mathrm{FF}}, \mathcal{O}_{X_{\mathrm{FF}}}(d+1)/t\mathcal{O}_{X_{\mathrm{FF}}}(d)) \cong P_{d+1}/tP_d \cong C$$

where the first isomorphism is since $H^1(X_{\mathrm{FF}}, \mathcal{O}_{X_{\mathrm{FF}}}) = 0$. Hence the coherent module $\mathcal{O}_{X_{\mathrm{FF}}}(d+1)/t\mathcal{O}_{X_{\mathrm{FF}}}(d)$ is equivalent to $i_{\infty *}C$. In fact, by [FF18, Thm. 6.5.2 (4)], the point $y \in |X_{\mathrm{FF}}|$ bijects with the 1–dimensional $\mathbb{Q}_p$–vector subspace spanned by t in P_1, and so it would be more functorial for us to rewrite the short exact sequence above as

$$0 \longrightarrow \mathcal{O}_{X_{\mathrm{FF}}}(d)\{1\} \longrightarrow \mathcal{O}_{X_{\mathrm{FF}}}(d+1) \longrightarrow i_{\infty *}C \longrightarrow 0 \tag{7.31}$$

viewed as coming from the inclusion $t \cdot P(d) \subseteq P(d+1)$ rather than as the injection of graded $\mathbb{Q}_p$–modules $t\colon P(d) \to P(d+1)$. Here, we have followed [FF18, Before Lem. 8.3.3] and denoted by $\mathcal{O}_{X_{\mathrm{FF}}}(d)\{1\}$ for the Tate twist $\mathcal{O}_{X_{\mathrm{FF}}}(d) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p\{t\}$, the coherent $\mathcal{O}_{X_{\mathrm{FF}}}$–module associated to $t \cdot P(d)$. For our purposes, we will be particularly interested in the case $d = 0$, which will allow us to build many modifications of vector bundles later. It should be mentioned that the fundamental line bundle sequence (7.31) is a geometrisation of the fundamental exact sequence of p–adic Hodge theory [FF18, Thm. 6.4.1].

Lemma 7.32 ([FF18, Lem. 8.3.3])**.** *The C–vector space* $\mathrm{Ext}^1(i_{\infty *}C, \mathcal{O}_{X_{\mathrm{FF}}})$ *is* 1*–dimensional generated by the extension*

$$0 \longrightarrow \mathcal{O}_{X_{\mathrm{FF}}}\{1\} \longrightarrow \mathcal{O}_{X_{\mathrm{FF}}}(1) \longrightarrow i_{\infty *}C \longrightarrow 0$$

from (7.31).

Proof. Since vector bundles are torsion–free and $i_{\infty *}C$ is a torsion sheaf, $\mathcal{H}\mathrm{om}(i_{\infty *}C, \mathcal{O}_{X_{\mathrm{FF}}}) = 0$, we get from the local–to–global Ext spectral sequence that

$$\mathrm{Ext}^1(i_{\infty *}C, \mathcal{O}_{X_{\mathrm{FF}}}) \cong H^0(X_{\mathrm{FF}}, \mathcal{E}\mathrm{xt}^1(i_{\infty *}C, \mathcal{O}_{X_{\mathrm{FF}}})).$$

Since sheafifications preserve stalks, we see that $\mathcal{E}\mathrm{xt}^1(i_{\infty *}C, \mathcal{O}_{X_{\mathrm{FF}}})$ is a skyscraper sheaf supported at $\infty \in X_{\mathrm{FF}}$, and this may be computed to be $i_{\infty *}C$ since locally around ∞, we may work affine locally with B_{dR}^+ and by the short exact sequence (where we have used that ξ is not a zero–divisor) coming from Fact 7.26 (2)

$$0 \longrightarrow B_{\mathrm{dR}}^+ \xrightarrow{\xi} B_{\mathrm{dR}}^+ \longrightarrow B_{\mathrm{dR}}^+/\xi = C \longrightarrow 0$$

we obtain that

$$C = B_{\mathrm{dR}}^+/\xi \cong \mathrm{coker}\left(\mathrm{Hom}_{B_{\mathrm{dR}}^+}(B_{\mathrm{dR}}^+, B_{\mathrm{dR}}^+) \xrightarrow{\xi} \mathrm{Hom}_{B_{\mathrm{dR}}^+}(B_{\mathrm{dR}}^+, B_{\mathrm{dR}}^+)\right) \xrightarrow{\cong} \mathrm{Ext}^1_{B_{\mathrm{dR}}^+}(C, B_{\mathrm{dR}}^+)$$

as required. □

Construction 7.33 (Universal modifications, [FF18, After Lem. 8.3.3])**.** Using Lemma 7.32, we see that for any $V \in \mathrm{Vect}^{\mathrm{fd}}_{\mathbb{Q}_p}$ and $W \in \mathrm{Vect}^{\mathrm{fd}}_{C}$, writing $V_C = V \otimes_{\mathbb{Q}_p} C$, we have an isomorphism

$$\mathrm{Hom}_C(W\{1\}, V) \cong \mathrm{Ext}^1(i_{\infty *}W, V \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathrm{FF}}}) \tag{7.34}$$

and unwinding the identifications, we see that this isomorphism is effected as follows: for a C–linear map $u\colon W\{1\} \to V_C$, we obtain an extension of $i_{\infty *}W$ by $V \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathrm{FF}}}$ given as the pullback in

$$\begin{tikzcd}
0 \arrow[r] & \underbrace{V\{-1\} \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathrm{FF}}}\{1\}}_{V \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathrm{FF}}}} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & i_{\infty *}W \arrow[r] \arrow[d, "i_{\infty *}u\{-1\}"] & 0 \\
0 \arrow[r] & \underbrace{V\{-1\} \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathrm{FF}}}\{1\}}_{V \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathrm{FF}}}} \arrow[r] & V\{-1\} \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathrm{FF}}}(1) \arrow[r] & \underbrace{V\{-1\} \otimes_{\mathbb{Q}_p} i_{\infty *}C}_{i_{\infty *}V_C\{-1\}} \arrow[r] & 0
\end{tikzcd}$$

where the bottom exact sequence is obtained by tensoring the fundamental sequence in Lemma 7.32 with $V\{-1\}$. By virtue of the isomorphism (7.34) and that $\mathcal{E}$ is locally free (i.e. a vector bundle) if and only if u is injective (c.f. [Sha19]), we see that *all* modifications (i.e. extensions of $i_{\infty *}W$ by $V \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathrm{FF}}}$) are obtained uniquely in this manner.

The next result is reminiscent of the usual finiteness result of vector spaces, namely that an injection of finite–dimensional vector spaces of the same dimension is an isomorphism.

Proposition 7.35 (Rank–degree detection, [SW13, Prop. 5.1.1])**.** *Let* $f\colon \mathcal{F} \to \mathcal{E}$ *be an injective map of coherent sheaves on* X_{FF} *of the same rank and degree. Then* f *is an isomorphism.*

The following is now the omnibus results we need for the construction of modifications of vector bundles.

Proposition 7.36 ([SW13, Prop. 5.1.6])**.** *Let G be a p–divisible group over $\mathcal{O}_C$. Let $\mathcal{E} = \mathcal{E}(G_0)$ for $G_0 = G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$ and $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{Z}_p} T$ be associated vector bundles over X, where we have written T for $T(G)(\mathcal{O}_C)$.*

(1) There is a natural exact sequence of coherent sheaves over X

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow i_{\infty *}(\operatorname{Lie} G \otimes C) \longrightarrow 0.$$

The global sections of this map give the logarithm sequence

$$0 \longrightarrow T[p^{-1}] \longrightarrow \widetilde{G}(\mathcal{O}_C) \longrightarrow \operatorname{Lie} G \otimes C \longrightarrow 0.$$

(2) Under the identification

$$i_\infty^* \mathcal{E} = M(G) \otimes_{\mathcal{O}_C} C,$$

the adjunction unit $\mathcal{E} \to i_{\infty *} i_\infty^* \mathcal{E}$ induces on global sections the quasi–logarithm map

$$\widetilde{G}(\mathcal{O}_C) \longrightarrow M(G) \otimes C.$$

When restricted to $T[\frac{1}{p}]$, it induces a surjective map

$$\alpha_G \colon T \otimes C \longrightarrow (\operatorname{Lie} G^\vee \otimes C)^\vee \subset M(G) \otimes C.$$

Proof. We only prove (1). By construction, $i_\infty^* \mathcal{E} \cong M(G) \otimes_{\mathcal{O}_C} C$. Hence, via the Hodge–de Rham (a.k.a. Grothendieck–Messing) map $M(G) \otimes_{\mathcal{O}_C} C \twoheadrightarrow \operatorname{Lie} G \otimes C$ and by adjunction, we obtain a surjection of sheaves $\mathcal{E} \to i_{\infty *}(\operatorname{Lie} G \otimes C)$. Write $\mathcal{F}' := \ker\big(\mathcal{E} \to i_{\infty *}(\operatorname{Lie} G \otimes C)\big)$ On the other hand, by the period map $T \to M[p^{-1}]^{\varphi = p}$ and the exact sequence from Corollary 7.28, we obtain a map $\mathcal{F} \to \mathcal{F}'$. We would like to show that this is an isomorphism.

Firstly, observe that $\mathcal{F}$ and $\mathcal{F}'$ have the same rank. It turns out that they also have equal degrees, namely 0, by an explicit computation using the classification of vector bundles on the curve (c.f. [FF18, Proof of Prop. 8.3.2]). Hence, by Proposition 7.35, if we can show that the map is injective, we would be done. And for this, it suffices to check that $\mathcal{F} \to \mathcal{E}$ is injective. For this, consider the commuting diagram

$$\begin{tikzcd}
 & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & i_{\infty *}(\operatorname{Lie} G \otimes C) \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{O}_{X_{\mathrm{FF}}} \otimes T\{-1\} \arrow[r] & i_{\infty *}(T \otimes C\{-1\}) \arrow[r] & 0
\end{tikzcd}$$

where the bottom sequence is obtained by tensoring $T(G)\{-1\}$ with the fundamental line bundle sequence (7.31) in the case $d = 0$. Since the bottom sequence is exact, in particular $\mathcal{F} \to \mathcal{O}_{X_{\mathrm{FF}}} \otimes T\{-1\}$ is injective, and so also $\mathcal{F} \to \mathcal{E}$ as wanted. ☐

Remark 7.37. In other words, Proposition 7.36 (1) says that a p–divisible group of height h over $\mathcal{O}_C$ gives rise to a modification $\mathcal{E}$ of the trivial vector bundle $\mathcal{F} = \mathcal{O}_X^h$ along $\infty \in X_{\mathrm{FF}}$.

7.4 The linear algebra classification theorem

7.4.1 Formal schemes and generic fibres

Fix a nonarchimedean field K with ring of definition $\mathcal{O} = \mathcal{O}_K$ and fix some $\varpi \in \mathcal{O}$, $|\varpi| < 1$. Consider the category $\mathrm{Nilp}_{\mathcal{O}}$ of $\mathcal{O}$–algebras R on which ϖ is nilpotent. Its opposite has the structure of a site given by the Zariski covers. We thus get the topos $\mathrm{Shv}(\mathrm{Nilp}_{\mathcal{O}}\mathrm{op})$.

As an example of a nonrepresentable sheaf, any adic $\mathcal{O}$–algebra A with ideal of definition I containing ϖ gives rise to the sheaf

$$(\mathrm{Spf} A)(R) := \operatorname*{colim}_n \mathrm{Hom}(A/I^n, R)$$

on $\mathrm{Nilp}_{\mathcal{O}}\mathrm{op}$. As before, one may define open embeddings in $\mathrm{Shv}(\mathrm{Nilp}_{\mathcal{O}}\mathrm{op})$ and hence define formal schemes over $\mathcal{O}$ as those sheaves on $\mathrm{Nilp}_{\mathcal{O}}\mathrm{op}$ admitting an open cover by $\mathrm{Spf} A$ for A as above.

The theory of adic spaces subsumes the theory of formal schemes in the following way.

Proposition 7.38 ([SW13, Prop. 2.2.1]). *The functor* $\mathrm{Spf} A \mapsto \mathrm{Spa}(A, A)$ *extends to a fully faithful functor* $(-)^{\mathrm{ad}} \colon \mathrm{LocFinFormalSch}_{\mathcal{O}} \hookrightarrow \mathrm{AdicSp}_{/\mathrm{Spa}(\mathcal{O},\mathcal{O})}$ *from formal schemes over* $\mathcal{O}$ *which locally admit a finitely generated ideal of definition to adic spaces over* $\mathrm{Spa}(\mathcal{O}, \mathcal{O})$*.*

Thus, given a formal scheme $\mathfrak{M}$, we may obtain its generic fibre (in an honest, direct way, as opposed to the world before adic spaces)

$$\mathfrak{M}^{\mathrm{ad}}_\eta := \mathfrak{M}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O},\mathcal{O})} \mathrm{Spa}(K, \mathcal{O}).$$

These types of objects always admit a moduli–theoretic description by virtue of the following:

Proposition 7.39 ([SW13, Prop. 2.2.2]). *Let* (R, R^+) *be a complete affinoid* $(K, \mathcal{O})$*–algebra.*

1. *The ring* R^+ *is the filtered union of its open and bounded* $\mathcal{O}$*–subalgebras* $R_0 \subset R^+$*,*
2. *The functor* $\mathfrak{M}^{\mathrm{ad}}_\eta \colon \mathrm{CAff}_{(K,\mathcal{O})} \to \mathrm{Set}$ *is the sheafification of*

$$(R, R^+) \mapsto \operatorname*{colim}_{R_0 \subset R^+} \mathfrak{M}(R_0) = \operatorname*{colim}_{R_0 \subset R^+} \lim_n \mathfrak{M}(R_0/\varpi^n).$$

In particular, we may define a functor

$$(-)^{\mathrm{ad}}_\eta \colon \mathrm{Shv}(\mathrm{Nilp}_{\mathcal{O}}\mathrm{op}) \longrightarrow \mathrm{Shv}(\mathrm{CAff}_{(K,\mathcal{O})}\mathrm{op})$$

taking a moduli problem over $\mathcal{O}$–algebras in which ϖ is nilpotent to a moduli problem on complete affinoid $(K, \mathcal{O})$–algebras by using the formula in the second part of the proposition above. Thus by construction, whenever an object in $\mathrm{Shv}(\mathrm{Nilp}_{\mathcal{O}}\mathrm{op})$ is representable by a formal scheme $\mathfrak{M}$ with locally finitely generated ideal of definition, then $\mathfrak{M}^{\mathrm{ad}}_\eta$ is representable by $\mathfrak{M}^{\mathrm{ad}}_\eta$ as constructed above using basechange.

In fact, this discussion extends to stacks. Namely, as an important example, consider the stack of p–divisible groups over $\mathrm{Nilp}_{\mathbb{Z}_p}\mathrm{op}$ which sends any R on which p is nilpotent to the groupoid of p–divisible groups over R with isomorphisms as morphisms. Observe that if A is an adic ring with ideal of definition I containing p, then giving a compatible system of p–divisible group over A/I^n for all $n \geq 1$ is equivalent to giving a p–divisible group over A. Using the previous construction, we get a stack on the category of adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which we still call the stack of p–divisible groups. Explicitly, for $X \in \mathrm{AdicSp}_{/\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}$, then giving a p–divisible group over X amounts to giving a cover of X by open subsets $U_i = \mathrm{Spa}(R_i, R_i^+) \subset X$, open and bounded $\mathbb{Z}_p$–subalgebras $R_{i,0} \subset R_i^+$, and p–divisible groups over $R_{i,0}$ satisfying the obvious compatibilities.

7.4.2 Period maps

Let H be a p–divisible group over a perfect field of characteristic p of height h and dimension d.

Definition 7.40. Let $R \in \mathrm{Nilp}_{W(k)}$. A deformation of H to R is a pair (G, ρ) where G is a p–divisible group over R and

$$\rho\colon H \otimes_k R/p \longrightarrow G \otimes_R R/p$$

is a quasi–isogeny. Let Def_H be the associated functor on $\mathrm{Nilp}_{W(k)}$ taking R to the set of isomorphism classes of deformations (G, ρ) of H to R.

Construction 7.41 (Rapoport–Zink spaces and period maps, [SW13, §6.1])**.** A theorem of Rapoport and Zink then shows that Def_H is representable by a formal scheme $\mathcal{M}$ over $\mathrm{Spf}\, W(k)$ which locally admits a finitely generated ideal of definition.

We now consider its associated generic fibre space $\mathcal{M}_\eta^{\mathrm{ad}}$, which is an adic space over $\mathrm{Spa}(W(k)[\frac{1}{p}], W(k))$. Now for any R a p–adically complete $W(k)$–algebra equipped with the p–adic topology and any $(G, \rho) \in \mathcal{M}_\eta^{\mathrm{ad}}(C, \mathcal{O}_C)$ the Grothendieck–Messing map gives rise to a quotient map

$$C^{\oplus h} \cong M(H) \otimes_{W(k)} C \cong MG(\mathcal{O}_C)_{\mathbb{Q}} \twoheadrightarrow \mathrm{Lie}(G_C)$$

to a d–dimensional C–vector space, which only depends on (G, ρ) up to isogeny. Writing $\mathcal{F}\ell$ for the flag variety parametrising d–dimensional quotients of the h–dimensional $W(k)[\frac{1}{p}]$–vector space $M(H)[\frac{1}{p}]$, which is defined as an adic space over $\mathrm{Spa}(W(k)[\frac{1}{p}], W(k))$, the construction above may be assembled to yield a map

$$\pi\colon \mathcal{M}_\eta^{\mathrm{ad}} \longrightarrow \mathcal{F}\ell$$

of adic spaces over $\mathrm{Spa}(W(k)[\frac{1}{p}], W(k))$ called the *Grothendieck–Messing period map.* This turns out to be an étale map and locally of finite type.

The period map is of course of great interest in general, but in this talk, the reason we will are interested in it is the following result, which allows us to glue our essential surjectivity result for spherically complete C with surjective norm map.

Theorem 7.42 ([SW13, Thm. 6.2.1])**.** *Fix a p–divisible group H over a perfect field k of dimension d and height h, and let $\mathcal{M}$ be the associated Rapoport–Zink space with period map*

$$\pi\colon \mathcal{M}_\eta^{\mathrm{ad}} \longrightarrow \mathcal{F}\ell.$$

Let C be an algebraically closed complete extension of $W(k)[p^{-1}]$ and let $x \in \mathcal{F}\ell(C, \mathcal{O}_C)$ be a point corresponding to a d–dimensional quotient $M(H) \otimes C \to W$. Let $\mathcal{E} = \mathcal{E}(H)$ be the vector bundle over X_{FF} associated to H, and consider the short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow i_{\infty *} W \longrightarrow 0 \tag{7.43}$$

corresponding to the quotient

$$E \longrightarrow i_{\infty *} i_\infty^* \mathcal{E} = i_{\infty *}(M(H) \otimes C) \longrightarrow i_{\infty *} W.$$

Then x is in the image of π if and only if $\mathcal{F}$ is isomorphic to $\mathcal{O}_{X_{\mathrm{FF}}}^h$.

Sketch proof. By Proposition 7.36 (1), a point in the image of π of course satisfies the triviality of $\mathcal{F}$. For the converse, since the map π is locally of finite type, we may reduce to the case when C is spherically complete with surjective norm map.

So suppose $\mathcal{F}$ is trivial and choose a $\mathbb{Z}_p$–lattice T in $H^0(X_{\mathrm{FF}}, \mathcal{F}) \cong \mathbb{Q}_p^{\oplus h}$. By the theory of modifications from Construction 7.33, we obtain an injection of C–vector spaces $u\colon W \rightarrowtail T \otimes_{\mathbb{Q}_p} C(-1)$ inducing the modification (7.43) via Construction 7.33. Thus, by the essential surjectivity in the special case of C being spherically complete with surjective norm map, there exists a p–divisible group G over $\mathcal{O}_C$ such that (7.43) is the one induced by G as in Proposition 7.36 (1).

Now, the identification $\mathcal{E}(G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p) = \mathcal{E} = \mathcal{E}(H) = \mathcal{E}(H \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p)$ together with the fully faithfulness statement in Theorem 7.29 (2) yields a quasi–isogeny

$$\rho\colon H \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p \dashrightarrow G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p,$$

thus giving a point $\widetilde{x} \in \mathcal{M}^{\mathrm{ad}}_\eta(C, \mathcal{O}_C)$ with $\pi(\widetilde{x}) = x$, as wanted. ☐

7.4.3 The “linear algebra” classification

Construction 7.44 (The linear algebra functor, [SW13, §5.2])**.** We write $p\text{–}\mathrm{Div}_{\mathcal{O}_C}$ for the category of p–divisible groups over $\mathcal{O}_C$ and $p\text{–}\mathrm{Latt}_C$ for the category of free $\mathbb{Z}_p$–modules T of finite rank equipped with a C–subvectorspace W of $T \otimes C(-1)$. We write the functor

$$T_{\mathrm{Lie}}\colon p\text{–}\mathrm{Div}_{\mathcal{O}_C} \longrightarrow p\text{–}\mathrm{Latt}_C$$

which sends a p–divisible group G to the tuple $(TG, W := \mathrm{Lie}\, G \otimes C)$, where we are viewing W as a C–subvectorspace of $T \otimes (-1)$ via the dual Hodge–Tate period map from Construction 7.19

$$\alpha^\vee_{G^\vee}\colon \mathrm{Lie}\, G \otimes C \longrightarrow T \otimes C(-1).$$

Proposition 7.45 (The reconstruction lemma, [Hon20, Prop. 3.4.9])**.** *Every p–divisible group G over $\mathcal{O}_K$ gives rise to a commutative diagram of exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \Phi_p G \arrow[r] \arrow[d] & G(\mathcal{O}_{C_K}) \arrow[r, "\log_G"] \arrow[d, "\alpha"] & t_G C_K \arrow[r] \arrow[d, "d\alpha"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mu_{p^\infty}(\overline{K})) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mu_{p^\infty}(\mathcal{O}_{C_K})) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), C_K) \arrow[r] & 0
\end{tikzcd}$$

where α and $d\alpha$ are Γ_K–equivariant and injective. In particular, we may functorially reconstruct $G(\mathcal{O}_{\mathbb{C}_K})$ from the linearised α map $d\alpha$ as the pullback as in the diagram.

Proof. Let us construct the map α and $d\alpha$. Consider

$$\begin{aligned}
T_p(G^\vee) = \lim_n G_n(\overline{K}) &\cong \lim_n G_n^\vee(\mathcal{O}_{C_K}) \\
&\cong \lim_n \mathrm{Hom}_{\mathcal{O}_{C_K}\mathrm{grp}}((G_n)_{\mathcal{O}_{C_K}}, (\mu_{p^n})_{\mathcal{O}_{C_K}}) \\
&= \mathrm{Hom}_{\mathrm{p-divgrp}}(G \times_{\mathcal{O}_K} \mathcal{O}_{C_K}, (\mu_{p^\infty})_{\mathcal{O}_K}).
\end{aligned}$$

We define α by setting

$$\alpha(g)(u) := u(g) \in \mu_{p^\infty}$$

where $g \in G$ and $u \in T_p(G^\vee)$, and $u(g)$ was defined using the identification of $T_p(G^\vee)$ above. We then define the map $d\alpha\colon t_G(C_K) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), C_K)$ by setting

$$d\alpha(z)(u) := du_{C_K}(z)$$

where $du_{C_K}\colon t_G(C_K) \longrightarrow t_{\mu_{p^\infty}}(C_K) \cong C_K$ is the map induced by u, again by the identification above. $\square$

Theorem 7.46 ([SW13, Thm. 5.2.1])**.** *The functor* $T_{\mathrm{Lie}}\colon p\text{–}\mathrm{Div}_{\mathcal{O}_C} \to p\text{–}\mathrm{Latt}_C$ *is an equivalence of categories.*

Sketch proof. By Proposition 7.45, we know that we may reconstruct the generic fibre G_η^{ad} of a p–divisible group G over $\mathcal{O}_C$ functorially as the pullback (functorial in G)

$$\begin{tikzcd}
G_\eta^{\mathrm{ad}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Lie}\, G \otimes C \arrow[d, "\alpha_G"] \\
T(G)(-1) \otimes_{\mathbb{Z}_p} \mu_{p^\infty} \arrow[r] & T(G)(-1) \otimes_{\mathbb{Z}_p} C.
\end{tikzcd}$$

We may then recover G as

$$G = \bigsqcup_{Y \subset G_\eta^{\mathrm{ad}}} H^0(Y; \mathcal{O}_Y^+)$$

where Y runs through the connected components of G_η^{ad}. Hence, any morphism between two p–divisible groups may be recovered from the map between their associated lattices, and vice versa. This gives fully faithfulness.

Next, for essential surjectivity, let $(T, W) \in p\text{–}\mathrm{Latt}_C$. By Construction 7.33, we may construct the associated modification of vector bundles on X_{FF}

$$0 \longrightarrow T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}} \longrightarrow \mathcal{E} \longrightarrow i_{\infty *} W \longrightarrow 0.$$

Since the fibre term $T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}}$ is trivial (and isomorphic to $\mathcal{O}_{X_{\mathrm{FF}}}^{\oplus h}$, we get from Theorem 7.42 a p–divisible group G over $\mathcal{O}_C$ inducing this lattice. $\square$

8 Moduli problems and some linear algebra (Vignesh Subramanian, 29 July)

8.1 Recollection on p-divisible groups

Let M be an R-module and G be a p-divisible group over R. Recall:

$$\mathrm{Ext}(G, \underline{M}) = \mathrm{Hom}_R(\omega_{G^\vee}, M)$$

Setting $M = \omega_{G^\vee}$, the identity element in the Hom set corresponds to the universal vectorial extension:

$$0 \longrightarrow \omega_{G^\vee} \longrightarrow E(G) \xrightarrow{\text{can}} G \longrightarrow 0. \tag{8.1}$$

Now consider a p-divisible group G over $\mathcal{O}_K$. Let G^0 be its identity component. This is necessarily a formal scheme $\mathrm{Spf}\, A^0$, where

$$A^0 = \mathcal{O}_K[\![t_1, \cdots, t_d]\!].$$

Let $I = \ker(A^0 \twoheadrightarrow \mathcal{O}_K)$ be the augmentation ideal. Then we define:

- $t_G M = \mathrm{Hom}_{\mathcal{O}_K}(I/I^2, M) = \mathrm{Lie}\, G(M)$.
- $M(G)$ is the Lie algebra of $E(G)$
- d is the dimension of G
- h is the height of G so that the rank of the finite flat group scheme G_n is p^{nh}.

Applying the Lie algebra functor to (8.1), we obtain:

$$0 \longrightarrow \omega_{G^\vee} \longrightarrow M(G) \xrightarrow{\log} \mathrm{Lie}\, G \longrightarrow 0.$$

Recall for a ring A with I-adic topology, the formal scheme of the pair (A, I) is defined to be

$$\mathrm{Spf}\, A := \operatorname*{colim}_n \mathrm{Spec}\, A/I^n.$$

In this sense, formal points on G are given by:

$$G(\mathcal{O}_K) := \varprojlim_n G(\mathcal{O}_K/\varpi^n).$$

A p-divisible group over an inverse system $\varprojlim\{X_i\}$ is a an inverse system $\{\mathscr{G}_i\}$ of p-divisible groups such that $\mathscr{G}_i \to X_i$ is a p-divisible group. Then given a p-divisible group $\mathscr{G}$ over an I-adic ring A, we obtain a system of p-divisible groups $\{\mathscr{G}_i\}$ via the pullback diagram:

NZ: Shouldn't this be colim of schemes?

$$\begin{tikzcd}
\mathscr{G}_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{G} \arrow[d] \\
\mathrm{Spec}\, A/I^n \arrow[r] & \mathrm{Spf}\, A.
\end{tikzcd}$$

Then $\Gamma(\{\mathscr{G}_i\}) = \mathscr{G}(\mathcal{O}_K) = \lim_n \mathscr{G}(A/I^n)$.

8.2 The Fargues–Fontaine curve

The Fargues–Fontaine curve is defined to to

$$X_{\mathrm{FF}} := \mathrm{Proj}\left(\bigoplus_{n\geq 0} (B^+_{\mathrm{cris}})^{\varphi = p^n}\right), \qquad \text{where } B^+_{\mathrm{cris}} := \widehat{\mathbf{A}_{\mathrm{inf}}\left[\frac{\xi^n}{n!}\right]}$$

and $\xi \in \mathbf{A}_{\mathrm{inf}}(\mathcal{O}_F)$ is the element that classifies the unique characteristic p untilt of F.

Let $\mathcal{E}$ be a vector bundle over X_{FF}. Then there exists a unique sequence of rational numbers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$ such that

$$\mathcal{E} \cong \bigoplus_{i=1}^m \mathcal{O}_{X_{\mathrm{FF}}}(\lambda_i).$$

Every vector bundle over X_{FF} corresponds to an isocrystal. The Frobenius φ acts on $B^+_{\mathrm{cris}}\langle e_1, e_2, \cdots, e_m\rangle$ by

$$e_1 \longmapsto e_2, \quad e_2 \longmapsto e_3, \quad \cdots, \quad e_{m-1} \longmapsto e_m, \quad e_m \longmapsto p^m e_1.$$

On top exterior form, φ acts on $\Lambda^m (B^+_{\mathrm{cris}})^{\oplus m}$ by multiplication by p^m. Conversely, we have a vector bundle interpretation of p-divisible groups over the special fiber.

Theorem 8.2 (Scholze–Weinstein)**.** *The assignment $G_0 \mapsto E(G_0)$ gives an equivalence of categories:*

or just 1-1 correspondence?

$$\{p\text{-divisible groups over } \mathcal{O}_C/p\}\,/_{\text{isogeny}} \longleftrightarrow \{\text{Vector bundles over } X_{\mathrm{FF}}\}$$

8.3 The Rapoport–Zink space

Let H be a p-divisible group of height h and dimension d over a perfect field k of characteristic p.

Definition 8.3. Let R be a nilpotent $W(k)$-algebra (i.e. $p^N R = 0$ for some N). A deformation of the p-divisible group H to R is a pair (G, ρ), where G is a p-divisible group over R and

$$\rho\colon H \otimes_k R/p \longrightarrow G \otimes_R R/p$$

is a quasi-isogeny.

Consider the functor

$$\begin{aligned} \mathrm{Def}_H\colon \mathrm{Nilp}^{\mathrm{op}}_{W(k)} &\longrightarrow \mathrm{Set}, \\ R &\longmapsto \{\text{deformations of } H \text{ to } R\}/\cong . \end{aligned}$$

Theorem 8.4 (Rapoport–Zink)**.** *The functor* Def_H *is represented by a formal scheme* $\mathcal{M}_H/\operatorname{Spf} W(k)$ *which locally admits a finitely generated ideal of definition. Moreover, all irreducible components of* $\mathcal{M}$ *are proper.*

Let $M(H)$ be the Dieudonné module of H. This is a free $W(k)$-module of rank h. A deformation $(G, \rho) \in \mathcal{M}$ of H then gives rise to a surjective map to a d-dimensional vector space:

$$M(H) \otimes R[1/p] \longrightarrow\!\!\!\!\to \operatorname{Lie} G[1/p].$$

This induces the Grothendieck–Messing period map:

$$\Pi_{GM}\colon (\mathcal{M}_H)^{\mathrm{ad}}_{\eta} \longrightarrow \mathcal{F}\ell_{h,d},$$

where $\mathcal{F}\ell_{h,d}$ classifies d-dimensional quotients of h-dimensional $W(k)[1/p]$-vector spaces. Next, we define the Rapoport–Zink space at infinite level.

Definition 8.5. Let $\mathcal{M}_n$ be a complete $(W(k)[1/p], W(k))$-algebra whose representable functor sends a Huber pair (R, R^+) to triples:

$$\left\{ (G,\rho,\alpha) \;\middle|\; \begin{array}{l} (G,\rho) \in (\mathcal{M}_H)^{\mathrm{ad}}_{\eta}(R,R^+) \text{ is a deformation,} \alpha\colon \underline{(\mathbb{Z}/p^n)^{\oplus h}} \to G[p^n]^{\mathrm{ad}}_{\eta}, \\ \text{such that for any closed point } x\colon \operatorname{Spa}(K,K^+) \to \operatorname{Spa}(R,R^+), \\ \alpha(x)\colon (\mathbb{Z}/p^n)^{\oplus h} \xrightarrow{\sim} G[p^n]^{\mathrm{ad}}_{\eta}(K,K^+) \text{ is an isomorphism.} \end{array} \right\}.$$

For $n = \infty$, we replace $G[p^n]$ and $(\mathbb{Z}/p^n)^{\oplus h}$ by the Tate module $T_p(G)$ of G and $\mathbb{Z}_p^{\oplus h}$, respectively.

Theorem 8.6 (Scholze–Weinstein)**.** *The Rapoport–Zink space* $\mathcal{M}_\infty$ *is represented by an adic space over* $\operatorname{Spa}(W(k)[1/p], W(k))$. *Moreover, we have an isomorphism of pre-perfectoid spaces*

$$\mathcal{M}_\infty \cong \varprojlim_n \mathcal{M}_n$$

Remark 8.7. When H is a connected p-divisible group over an algebraically field $k = \bar{k}$ of dimension 1 and height $h < \infty$, the infinite level Rapoport–Zink space $\mathcal{M}^h_\infty$ is the Lubin–Tate space at infinite level.

8.4 EL structures

The general duality isomorphism proven in [SW13] is an equivalence between moduli problems for p-divisible groups with additional structures. The following definition is from [RZ96]:

IM: add citation

IM: I've merged the definitions in this and the next talk, to avoid repetition.

Definition 8.8. A *rational EL-datum* is a tuple

$$\mathcal{D} = (B, V, \widetilde{H}, \mu)$$

consisting of

1. a semisimple $\mathbb{Q}_p$-algebra B with maximal order $\mathcal{O}_B$,
2. a B-module V of rank d,
3. a morphism $\mu\colon \mathbb{G}_m \to \mathcal{G}_{\overline{\mathbb{Q}}_p}$ such that the induced grading on $V_{\overline{\mathbb{Q}}_p}$ admits a two-term decomposition
$$V_{\overline{\mathbb{Q}}_p} \cong V_0 \oplus V_1$$
with V_0 of rank h,
4. a formal $\mathcal{O}_B$-module H over k of dimension d and height h such that
$$MH \otimes_{\mathcal{O}} K \cong V \otimes_{\mathbb{Q}_p} K.$$
We write $\tilde{H}$ for its adic generic fibre

IM: Check

Here $\mathcal{G}$ denotes the algebraic group $\mathrm{GL}_B(V)$. The associated *integral EL-datum* is

$$\mathcal{D}^{\mathrm{int}} = (\mathcal{O}_B, \Lambda, H, \mu),$$

where Λ is an $\mathcal{O}_B$-latice in V.

Remark 8.9. These determine moduli problems with prescribed *e*ndomorphisms and *l*evel structure. The motivation comes from Drinfeld's moduli problem, and we'll discuss in Talk 9.

To every EL-datum one can associate a moduli problem on nilpotent $\mathcal{O}$-algebras by

$$\mathcal{M}^{\mathcal{D}}(R) = \{(G, \rho\colon H \otimes_k R/p \to G \otimes_R R/p)\}$$

where G has an $\mathcal{O}_B$-module structure and ρ is an $\mathcal{O}_B$-linear isomorphism. We saw in Talk 7 that this moduli problem has a linear algebra description: it is the sheafiification of the functor that sends

$$(R, R^+) \longmapsto \{B\text{-module maps } V \longrightarrow \tilde{H}(R, R^+) \text{ such that } \ldots\}$$

As such, we can also define an infinite-level moduli problem $\mathcal{M}^{\mathcal{D}}_\infty$ by sheafifying the functor

$$(R, R^+) \longmapsto \{B\text{-module maps } V \longrightarrow \tilde{H}(R, R^+) \text{ such that } \ldots\}$$

where the target contains maps satisfy:

1. $W = M(H) \otimes_{W(k)} R/V \otimes R$ is a finitely generated project R-module

2. For any closed point $x\colon \operatorname{Spa}(C, \mathcal{O}_C) \to \operatorname{Spa}(R, R^+)$, we have a short exact sequence:
$$0 \longrightarrow V \longrightarrow \tilde{H}^{\mathrm{ad}}_\eta(C, \mathcal{O}_C) \longrightarrow W \otimes_R C \longrightarrow 0.$$

NZ: geometric point??

Dualizing the above, we define:

$$\mathcal{M}_\infty^{\check{\mathcal{D}}} = \{\mathrm{End}_B(V)\text{-linear maps } \check{s}\colon \mathrm{End}_B^0(H) \to V^* \otimes_B \tilde{H}_\eta^{\mathrm{ad}}(R, R^+).\}$$

In fact, the infinite-level moduli problem classifies p-divisible with EL-structure, and this duality arises from a duality $\mathcal{D} \mapsto \check{\mathcal{D}}$ at the level of EL-data defined in an explicit algebraic way; we'll say a little more on this next time.

Theorem 8.10 ([SW13], Theorem ??). *There a $\mathcal{G} \times \check{\mathcal{G}}$-equivariant equivalence:*

cite

$$\mathcal{M}_\infty^{\mathcal{D}} \simeq \mathcal{M}_\infty^{\check{\mathcal{D}}}.$$

9 Proof Outline, Lazard's theorem and Galois cohomology of $\mathcal{O}_C$ (Itamar Mor, 1 August)

9.1 A bit more on EL-structures

We give a bit of context to the definition of EL-structures, which at first glance seems somewhat esoteric. We begin with our favourite moduli problem: let H_h be a one-dimensional formal group law of height h over an algebraic extension k of $\mathbb{F}_p$. This gives rise to the Lubin-Tate moduli problem on $\mathrm{Nilp}_{W(k)}$,

$$R \mapsto \mathrm{LT}(R) = \{(G \text{ formal group over } R,\ \rho\colon H \otimes_k R/p \xrightarrow{\sim} G \otimes_R R/p)\}.$$

Recall that the Morava stabiliser group $\mathbb{G}_h$ is defined by

$$\mathbb{G}_h = \mathrm{Aut}(\overline{k}, H_{\overline{k}}).$$

In particular, the latter contains a class Π representing the Frobenius automorphism of H and

$$\mathrm{End}(H)_{k_h} \cong W(k_h)\langle \Pi \rangle / \Pi^h - p$$

if k_h/k is a finite extension with Galois group $\mathbb{Z}/h$ The notation denotes a skew-commutative polynomial algebra: for all $a \in W(k_h)$ we have the relation

$$\Pi a = \mathrm{Frob}(a)\Pi.$$

Recall that one can further identify this ring with the maximal order $\mathcal{O}_D \subset D$ in the division algebra $D/K := W(k)[1/p]$ classified by the invariant

$$1/h \in \mathrm{Br}(K) \cong \mathbb{Q}/\mathbb{Z}.$$

Remark 9.1. The Lubin-Tate moduli problem can be refined by adding in *level structures*, i.e. trivialisations of the p^n-torsion: one obtains a tower of moduli problems

$$\cdots \to \mathcal{M}_n^{\mathrm{LT}} \to \cdots \to \mathcal{M}_1^{\mathrm{LT}} \to \mathcal{M}_0^{\mathrm{LT}}$$

on $\mathrm{Nilp}_{W(k)}$ defined by

$$\mathcal{M}_n^{\mathrm{LT}}(R) = \{(G, \rho, \alpha\colon (\mathbb{Z}/p^n)^h \xrightarrow{\sim} G[p^n](R))\}$$

with $(G, \rho) \in \mathcal{M}_0^{\mathrm{LT}}(R)$. An essential observation is that these moduli problems with level structures admit further group actions. Indeed, the action of $\mathrm{GL}_h(\mathbb{Z}/p^n)$ on $(\mathbb{Z}/p^n)^h$ induces an action on $\mathcal{M}_n^{\mathrm{LT}}$ that is compatible with the reduction maps in the tower above. In the limit we obtain an action of $\mathrm{GL}_n(\mathbb{Z}_p)$.

To make sense of this limit, we follow [SW13] in implicitly taking the adic generic fibre everywhere (i.e. basechange to $\operatorname{Spa} K$) .

Definition 9.2. Define $\mathcal{M}_\infty^{\mathrm{LT}}$ to be the limit of the tower $\{\mathcal{M}_{n,K}^{\mathrm{LT}}\}_n$, where this limit should be taken in pre-perfectoid spaces[10]. By the remark above, this defines a $\mathrm{GL}_h(\mathbb{Z}_p)$-torsor

$$\mathcal{M}_\infty^{\mathrm{LT}} \to \mathcal{M}_K^{\mathrm{LT}}.$$

9.1.1 Drinfeld's moduli problem

We will now introduce a similar but *dual* moduli problem considered by Drinfeld [Dri76]. In the following, we let $\mathcal{O}$ be shorthand for $\mathcal{O}_K$.

Definition 9.3.

- A *formal $\mathcal{O}$-module* over R is a formal group G/R with a ring map

  $$i\colon \mathcal{O} \to \operatorname{End}_R(G)$$

  such that i induces the standard action[11] on $\operatorname{Lie}(G)$.
- A *formal $\mathcal{O}_D$-module* over R is a formal $\mathcal{O}$-module (G, i) with a factorisation of rings

  $$\begin{tikzcd} & \mathcal{O} \arrow[dl] \arrow[dr, "i"] & \\ \mathcal{O}_D \arrow[rr, dotted, "a"] & & \operatorname{End}_R(G). \end{tikzcd}$$

- A formal $\mathcal{O}_D$-module (G, i) is said to be *special* if at every geometric point, the action on $\operatorname{Lie}(G)$ has all eigenspaces one-dimensional.

Definition 9.4. Fix a special formal $\mathcal{O}_D$-module H over $\mathcal{O}/p = k$ with dimension h and height h^2. Then for $R \in \mathrm{Nilp}_{\mathcal{O}}$ define

$$\mathcal{M}^{\mathrm{Dr}}(R) = \{(G, \rho\colon H \otimes_k R/p \to G \otimes_R R/p)\}$$

for G a special formal $\mathcal{O}_D$-module over R and ρ a $\mathcal{O}_D$-linear quasi-isogeny.

Remark 9.5. Note that there is now a natural action by precomposition of $\operatorname{Aut}_{\mathcal{O}_D}(H) \otimes \mathbb{Q}$ on $\mathcal{M}^{\mathrm{Dr}}(R)$.

Remark 9.6. In the definition we have made a choice of a special formal $\mathcal{O}_D$-module over k with prescribed dimension and height. However, Drinfeld proves that there is a unique such object over $\overline{k}$, as in the Lubin-Tate case; denote this by H_h^{Dr}. In fact,

$$H_h^{\mathrm{Dr}} = H_h \oplus H_h^{(1)} \oplus \cdots \oplus H_h^{(h-1)}.$$

Furthermore, its endomorphism ring can be identified as

$$\operatorname{End}_{\mathcal{O}_D}(H_h^{\mathrm{Dr}}) \cong M_{h\times h}(\mathbb{Z}_p),$$

a matrix ring over $\mathbb{Z}_p$—this follows from a computation of the endomorphisms of its Dieudonné module. For details see [RZ96, § 3.54-78] or [FGL08, § II.2.3-4].

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[10] These are similar to perfectoid spaces but do not require the sheafy condition on their affinoid perfectoid building blocks.

[11] If one equips G with a coordinate, this requires that the action of $\mathcal{O}$ is just left multiplication up to second order terms

Theorem 9.7 ([Dri76]). *For any choice H of a special formal $\mathcal{O}_D$-module over k as above, the moduli problem $\mathcal{M}^{\mathrm{Dr}}$ is representable by a formal scheme $\widehat{\Omega}$ over $\mathcal{O}$. Furthermore, its generic fibre is given by*

$$\mathcal{H} := \widehat{\Omega}_K = \mathbb{P}^{h-1}_{\mathbb{Q}_p} \setminus \mathbb{P}^{h-1}(\mathbb{Q}_p).$$

Once again, we can define an analogue of this moduli problem with level structures by

$$\mathcal{M}^{\mathrm{Dr}}_n(R) = \{(G, \rho, \alpha\colon \mathcal{O}_D/p^n \xrightarrow{\sim} G[p^n](R))\}$$

where α is now an $\mathcal{O}_D$-linear isomorphism. This moduli problem admits an action of $\mathrm{Aut}_{\mathcal{O}_D}(\mathcal{O}_D/p^n)$ by precomposition, so we obtain a $\mathcal{O}_D^\times$-torsor of pre-perfectoid spaces

$$\mathcal{M}^{\mathrm{Dr}}_\infty := \varprojlim_n \mathcal{M}^{\mathrm{Dr}}_{n,K} \to \mathcal{M}^{\mathrm{Dr}}_K.$$

9.1.2 EL-data and duality

The analogous construction of the two moduli problems above leads us to develop a common abstraction of both in the form of EL-data. Recall Definition 8.8 that these are tuples

$$\mathcal{D} = (B, V, \tilde{H}, \mu) \quad \text{and} \quad \mathcal{D}^{\mathrm{int}} = (\mathcal{O}_B, \Lambda, H, \mu),$$

with associated moduli problem that we denoted $\mathcal{M}^{\mathcal{D}}$. The version with level structures is defined analogously as

$$\mathcal{M}^{\mathcal{D}}_n(R) = \{(G, \rho, \alpha\colon \Lambda/p^n \to G[p^n](R)) \mid \cdots\}$$

where α is now an $\mathcal{O}_B$-linear isomorphism satisfying some conditions that we will elide for brevity: see [RZ96, Definition 3.21]. This formalism recovers the moduli problems discussed above:

- The integral EL-datum $\mathcal{D}^{\mathrm{int}}_{\mathrm{LT}} = (\mathbb{Z}_p, \mathbb{Z}_p^h, H_h, \mu)$ recovers[12]

  $$\mathcal{G} = \mathrm{GL}_h(\mathbb{Q}_p), \quad \mathcal{J} = D^\times, \quad \mathcal{M}^{\mathcal{D}_{\mathrm{LT}}} = \mathcal{M}^{\mathrm{LT}}.$$

  In this case, the conditions are empty.

- The integral EL-datum $\mathcal{D}^{\mathrm{int}}_{\mathrm{Dr}} = (\mathcal{O}_D, \mathcal{O}_D, H_h^{\mathrm{Dr}}, \mu)$ recovers

  $$\mathcal{G} = D^\times, \quad \mathcal{J} = \mathrm{GL}_h(\mathbb{Q}_p), \quad \mathcal{M}^{\mathcal{D}_{\mathrm{Dr}}} = \mathcal{M}^{\mathrm{Dr}}.$$

  In this case, the conditions amount to the requirement that G is *special*.

Remark 9.8. This moduli problem admits an action of the $\mathbb{Q}_p$-points of

$$\mathcal{J} := (\mathrm{End}_{\mathcal{O}_B}(H) \otimes \mathbb{Q})^\times$$

by precomposition, and the version with level structures admits an action of $\mathrm{Aut}_{\mathcal{O}_B}(\Lambda/p^n)$. Once again, we can take the limit of the tower of level structures in preperfectoid spaces to obtain an action of $\mathrm{Aut}_B(V) = \mathcal{G}$ on $\mathcal{M}^{\mathcal{D}}_\infty$ (again, since we allow isogenies).

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[12] More precisely, this recovers the version of Lubin-Tate space where we allow deformations up to isogeny. In particular, $\mathcal{J} \cong \mathbb{G}_h \times \mathbb{Z}$.

There is a straightforward way to define, in terms of linear algebra, the *dual* $\check{\mathcal{D}}$ of an EL-datum $\mathcal{D}$ [SW13, §7.2]; this duality is defined so as to interchange the roles of $\mathcal{G}$ and $\mathcal{J}$, in these sense that

$$\check{\mathcal{G}} = \mathcal{J} \quad \text{and} \quad \check{\mathcal{J}} = \mathcal{G}.$$

Concretely, one sets

$$\check{B} = \mathrm{End}_B(H) \otimes \mathbb{Q} \quad \text{and} \quad \check{V} = \check{B}.$$

The group $\check{H}$ is determined by its Dieudonné module, which we set to be $M\check{H} := \Lambda^\vee \otimes_{\mathcal{O}_B} MH$—here $\Lambda^\vee$ is the $\mathcal{O}_B$-linear dual. The cocharacter $\check{\mu}$ arises naturally from the filtration determined by μ. The twin towers isomorphism can then be placed in the general context of duality on Rapoport-Zink spaces associated to dual EL-data:

Theorem 9.9 ([SW13], Theorem E)**.** *There is an $\mathcal{G} \times \mathcal{J}$-equivariant equivalence of pre-perfectoid spaces*

$$\mathcal{M}_\infty^{\check{\mathcal{D}}} \simeq \mathcal{M}_\infty^{\mathcal{D}}.$$

The equality $\check{\mathcal{D}}_{\mathrm{LT}} = \mathcal{D}_{\mathrm{Dr}}$ is really the prototypical example of this duality, and this makes precise in which sense these moduli problems are dual.

9.2 Back to [Bar+24]

Let us now return to the main theorem of the paper and how the ingredients we will discuss in the following lectures feature in the proof. The goal is to prove the following:

Theorem 9.10 ([Bar+24], Theorem A)**.** *There is an isomorphism*

$$\pi_*(L_{K(h)}\mathbb{S} \otimes \mathbb{Q}) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \ldots, x_{2h-1})$$

We saw in the first few talks that this follows from the collapse of the rational $K(h)$-local E_h-Adams spectral sequence, and in particular from the identification of its E_2-page $H^s(\mathbb{G}_h, A[u_h^{\pm 1}])$. For $t \neq 0$ the cohomology $H^s(\mathbb{G}_h, A\{u_h^t\})$ is zero for easy reasons [Bar+24, Lemma 2.6.1]—in fact, the cohomology of $\mathbb{Z}_p \subset Z(\mathcal{O}_D^\times)$ is already zero. Thus it remains to prove:

Theorem 9.11 ([Bar+24], Theorem B)**.** *The equivariant inclusion $W \hookrightarrow A$ induces an isomorphism*

$$H^*_{\mathrm{cont}}(\mathbb{G}_h; A \otimes \mathbb{Q}_p) \cong H^*_{\mathrm{cont}}(\mathbb{G}_h; W \otimes \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \ldots, x_{2h-1}),$$

the second isomorphism being Lazard's theorem.

Broadly, the idea is to convert Theorem B into a problem about simpler cohomology using the twin towers principle outlined above. Indeed, Talks 4 to 8 provided us with a diagram of pre-perfectoid spaces of the form

$$\begin{tikzcd}
 & \mathcal{M}_\infty^{\mathrm{LT}} \simeq \mathcal{M}_\infty^{\mathrm{Dr}} \arrow[dl, "\mathrm{GL}_h(\mathbb{Z}_p)"'] \arrow[dr, "\mathbb{G}_h"] & \\
\mathrm{LT}_K = \mathcal{M}_K^{\mathrm{LT}} & & \mathcal{M}_K^{\mathrm{Dr}} = \mathcal{H}.
\end{tikzcd} \tag{9.12}$$

in which each arrow is a torsor for the displayed group. In particular, we obtain an equivalence of quotient stacks

$$[\mathrm{LT}_K/\mathbb{G}_h] \simeq [\mathcal{M}_\infty^{\mathrm{LT}}/(\mathbb{G}_h \times \mathrm{GL}_h(\mathbb{Z}_p))] \simeq [\mathcal{M}_\infty^{\mathrm{Dr}}/(\mathbb{G}_h \times \mathrm{GL}_h(\mathbb{Z}_p))] \simeq [\mathcal{H}/\mathrm{GL}_h(\mathbb{Z}_p)].$$

This is already quite useful, since it allows us to replace the cohomology of $\mathbb{G}_h$ with the cohomology of $\mathrm{GL}_h(\mathbb{Z}_p)$. Indeed, on global sections we obtain

$$\mathbf{R}\Gamma(\mathbb{G}_h; \mathbf{R}\Gamma(\mathrm{LT}_K; \widehat{\mathcal{O}}^+_{\mathrm{LT}_K})) \simeq \mathbf{R}\Gamma(\mathbb{G}_h \times \mathrm{GL}_h(\mathbb{Z}_p); \mathbf{R}\Gamma(\mathcal{M}^{\mathrm{LT}}_\infty; \widehat{\mathcal{O}}^+_{\mathcal{M}^{\mathrm{LT}}_\infty})) \simeq \mathbf{R}\Gamma(\mathrm{GL}_h(\mathbb{Z}_p); \mathbf{R}\Gamma(\mathcal{H}; \widehat{\mathcal{O}}^+_{\mathcal{H}})).$$

On the other hand, there is still some work to do: the left hand side of this equivalence does *not* compute the target of Theorem B. Indeed, a choice of coordinates shows that

$$\mathrm{LT}_K \simeq \mathring{\mathbb{D}}^{h-1}_K$$

is an adic open unit disc, and in particular is not affinoid perfectoid; hence we can not identify $\mathbf{R}\Gamma(\mathrm{LT}_K; \widehat{\mathcal{O}}^+_{\mathrm{LT}_K})$ with the Lubin-Tate ring A. To remedy this, the strategy proceeds in four main steps.

1. We construct comparison maps

$$\begin{aligned} \alpha_{\mathrm{LT}} \colon A[\epsilon] &\to \mathbf{R}\Gamma_{\text{proét}}(\mathrm{LT}_K; \widehat{\mathcal{O}}^+), \\ \alpha_{\mathcal{H}} \colon W[\epsilon] &\to \mathbf{R}\Gamma_{\text{proét}}(\mathcal{H}; \widehat{\mathcal{O}}^+) \end{aligned}$$

   where $R[\epsilon]$ stands for the two term complex $\underline{A} \xrightarrow{0} \underline{A}$. These will induce a commutative diagram

$$\begin{tikzcd} W[\epsilon]^{h\mathrm{GL}_h(\mathbb{Z}_p)} \arrow[r] \arrow[d, "\alpha_{\mathcal{H}}"'] & A[\epsilon]^{h\mathbb{G}_h} \arrow[d, "\alpha_{\mathrm{LT}_K}"] \\ \mathbf{R}\Gamma_{\text{proét}}(\mathcal{H}; \widehat{\mathcal{O}}^+)^{h\mathrm{GL}_h(\mathbb{Z}_p)} \arrow[r, "\sim"] & \mathbf{R}\Gamma_{\text{proét}}(\mathrm{LT}_K; \widehat{\mathcal{O}}^+)^{h\mathbb{G}_h} \end{tikzcd}$$

   where the bottom horizontal arrow is an equivalence by (9.12). In fact, these comparison maps α can be constructed more generally for any adic space $\mathcal{Y}$ with a suitable covering $\mathcal{U} = \{\mathfrak{Y}_{i,K}\}$ by the generic fibres of sufficiently nice formal schemes $\mathfrak{Y}_i$, and are in general of the form

$$\alpha_{\mathcal{Y}} \colon \check{C}^*(\mathcal{U}; \mathcal{O}) \otimes \mathcal{O}_K[\epsilon] \to \mathbf{R}\Gamma_{\text{proét}}(\mathcal{Y}; \widehat{\mathcal{O}}^+_{\mathcal{Y}}).$$

2. In the case where $\mathcal{Y} = \mathrm{Spa}(K, \mathcal{O}_K)$, the map takes the form

$$\alpha_{\mathcal{Y}} \colon \mathcal{O}_K[\epsilon] \to \mathbf{R}\Gamma_{\text{proét}}(\mathrm{Spa}(K); \widehat{\mathcal{O}}^+),$$

   where the right hand side computes the Galois cohomology

$$\pi_{-*}\mathbf{R}\Gamma_{\text{proét}}(\mathrm{Spa}(K); \widehat{\mathcal{O}}^+) \cong H^*_{\mathrm{cont}}(\Gamma; \mathcal{O}_C)$$

   for $\Gamma = \mathrm{Gal}(\overline{K}/K)$ and $C = \widehat{\overline{K}}$. We will see that $\alpha_{\mathcal{Y}}$ is a rational isomorphism by computing the Galois cohomology of $\mathcal{O}_C$.
3. For general $\mathcal{Y}$, we use Step 2. and descent to prove that $H^*(\mathrm{cof}(\alpha_{\mathcal{Y}}))$ is p-power torsion.
4. We conclude by computing

$$\check{C}^*(\mathcal{U}; \widehat{\mathcal{O}}^+_{\mathrm{LT}_K}) \cong A \quad \text{and} \quad \check{C}^*(\mathcal{U}; \widehat{\mathcal{O}}^+_{\mathcal{H}}) \cong W,$$

   which gives us the rational isomorphism

$$H^*_{\mathrm{cont}}(\mathbb{G}_h; A[\epsilon] \otimes \mathbb{Q}_p) \cong H^*_{\mathrm{cont}}(\mathrm{GL}_h(\mathbb{Z}_p); \mathbb{Q}_p[\epsilon]).$$

   Lazard's theorem on the rational cohomology of p-adic analytic Lie groups Lemma 1.5 tells us that there are isomorphisms

$$\begin{aligned} H^*_{\mathrm{cont}}(\mathrm{GL}_h(\mathbb{Z}_p); \mathbb{Q}_p) &\cong \Lambda_{\mathbb{Q}_p}(x_1, x_2, \ldots, x_{2h-1}), \\ H^*_{\mathrm{cont}}(\mathbb{G}_h; K) &\cong \Lambda_{\mathbb{Q}_p}(x_1, x_2, \ldots, x_{2h-1}). \end{aligned}$$

Therefore, we can count dimensions to obtain

$$H^*_{\mathrm{cont}}(\mathbb{G}_h; A \otimes \mathbb{Q}_p) \cong H^*_{\mathrm{cont}}(\mathbb{G}_h; \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_2, \ldots, x_{2h-1}),$$

which is the statement of Theorem B.

These steps will occupy the remaining talks: specifically, today and in Talk 10 we prove 2., in Talk 11 we discuss steps 1. and 3., and we conclude by proving 4. in Talk 12.

9.2.1 Galois cohomology of $\mathcal{O}_C$

Let's switch gears and discuss the cohomology computations needed in the plan above. We fix the same notation $C = \widehat{\overline{K}}$, $\Gamma = \mathrm{Gal}(\overline{K}/K)$. The goal of this section is to show that there is a rational equivalence

$$\mathbf{R}\Gamma(\Gamma; \mathcal{O}_C) \otimes_{\mathcal{O}_K} K \simeq \mathcal{O}_K[\epsilon] \otimes_{\mathcal{O}_K} K,$$

whence in particular the rational cohomology of Γ with coefficients in $\mathcal{O}_C$ is (almost) concentrated in degrees zero and one.

Remark 9.13. This should be surprising: indeed

$$H^1(\Gamma; \overline{K}) = 0,$$

so it is interesting that the first cohomology group with coefficients in the *completion* of the algebraic closure of K does not vanish.

Remark 9.14. In fact, much of the hard work in [Bar+24, §4] goes proving an *integral* version of Talk 9.2.1, which holds up to an explicit p-power torsion. We'll content ourselves with the rational statement, which is due to Tate.

The idea behind proving this goes back to Tate and relies on breaking up the extension $K \to C$ into an intermediate extension

$$K \xrightarrow{\mathbb{Z}_p} \widehat{K}_\infty \xrightarrow{\Gamma_\infty} C$$

The first extension has computable Galois cohomology (in particular, concentrated in degrees zero and one), while the second extension has cohomology (almost) concentrated in degree zero, whence we will be able to conclude.

In the rest of this talk we discuss the second statement, i.e. that $H^*(\Gamma_\infty; \mathcal{O}_C) \cong^a 0$ in positive degrees. This relies on a trick using traces in cohomology, which we sketch in greater generality. Let K be a finite extension of $\mathbb{Q}_p$ and L/K a finite G-Galois extension. We with a particular proof of the classical fact that

$$H^*(G; L) = 0$$

in positive degrees. For an element $y \in L$ and a cochain $f \in C^{n+1}(G; L)$ in the reduced standard complex, Tate defines the *cup product* $y \smile f \in C^n(G; L)$ by

$$y \smile f(g_1, \ldots, g_n) \coloneqq \sum_{g \in G} {}^{g_1 \cdots g_n g} y \cdot f(g_1, \ldots, g_n, g),$$

where the superscript denoted the Galois action on L.

Exercise 9.15. This satisfies the Leibniz-type formula

$$d(y \smile f) = \mathrm{Tr}_{L/K}(y) \cdot f + y \smile d(f).$$

Remark 9.16. In particular, if $f \in Z^{n+1}(G;L) \subset C^{n+1}(G;L)$ was a cocycle, the formula above gives us

$$d(y \smile f) = \mathrm{Tr}_{L/K}(y) \cdot f.$$

So if y has nonzero trace, we have written f as a coboundary $f = d(\mathrm{Tr}_{L/K}(y)^{-1} y \smile f)$ so that $[f] = 0$ in $H^*(G;L)$. In particular, if we such a y exists we may conclude that the higher Galois cohomology of L vanishes.

Next, we try to extend this trick to the infinite extension C/K. For$f \in C^*(\Gamma; C)$ let us write

$$f = p^{-j} \lim_n f_n \quad \text{for} \quad f_n \in C^*(\mathrm{Gal}(L_n/K); \mathcal{O}_{L_n}/p^n),$$

using the isomorphism

$$C^*(\Gamma; \mathcal{O}_C) \cong \varprojlim_n \varinjlim_{L/K} C^*(\mathrm{Gal}(L/K); \mathcal{O}_L/p^n).$$

where the colimit ranges over finite extensions: that is, we choose L_n/K such that f/p^n is defined over $\mathcal{O}_{L_n}/p^n$. For each of these finite extensions we can apply the trick above to trivialise the higher cohomology groups; this requires that we choose for each n some $y_n \in L_n$ with $\mathrm{Tr}_{L_n/K}(y_n) \neq 0$. To proceed, suppose we can choose y_n so that for fixed $x \in K^\times$ and c,

$$|y_n| \leq c \quad \text{and} \quad \mathrm{Tr}_{L_n/K}(y_n) \xrightarrow{n\to\infty} x. \tag{9.17}$$

We assume without loss of generality that each $y_n \in \mathcal{O}_{L_n}$, so also $x \in \mathcal{O}_K^\times$. The claim is that this will allow us to extend the trick to C. Write $G_n := \mathrm{Gal}(L_n/K)$.

Lemma 9.18. *Suppose that (9.17) holds. If $f'_n \in C^*(G_n, \mathcal{O}_{L_n})$ with $f'_n \to 0$, then*

$$y_n \smile_{L_n} f'_n \to 0 \in C^{*-1}(\Gamma, \mathcal{O}_C).$$

Proof. Indeed,

$$|y_n \smile_{L_n} f'_n(g_1, \ldots, g_k)| \leq \max_{g \in G_n} |{}^{g_1 \cdots g_k g} y_n||f'_n(g_1, \ldots, g_k, g)| \leq |f'_n| \to 0.$$ □

In particular, by continuity we obtain

$$\begin{aligned}\lim_n d(y_n \smile_{L_n} f_n) &= \lim_n \left[\mathrm{Tr}_{L_n/K}(y_n) f_n + y_n \smile_{L_n} df_n\right] \\ &= p^j (xf + \lim_n y_n \smile_{L_n} df_n).\end{aligned} \tag{9.19}$$

If f is a cocycle then the second term is zero by Lemma 9.18; if the $y_n \smile_{L_n} f_n$ converge we have therefore exhibited f as a cocycle, so we win. On the other hand, convergence of $y_n \smile_{L_n} f_n$ does not follow from convergence of f_n, since the $y_n \smile_{L_n} (-)$ are not compatible. To get around this, consider

$$f'_n := y_n \smile_{L_n} f_n - y_{n-1} \smile_{L_{n-1}} f_{n-1}.$$

Note that if f is a cycle then also $df'_n \to 0$, so that $y_n \smile_{L_n} df'_n \to 0$, again by Lemma 9.18. By Leibniz, we can therefore consider the Cauchy sequence

$$\sum_{i=2}^{\infty} (xf'_i - u_i d(y_i \smile_{L_i} f'_i))$$

with $u_i := \mathrm{Tr}_{L_i/K}(y_i)^{-1} x \in \mathcal{O}_K^\times$ (at least after throwing away finitely many f_n). This converges to some $f' \in C^{*-1}(\Gamma, \mathcal{O}_C)$, and up to a boundary each partial sum is of the form

$$x(y_n \smile_{L_n} f_n - y_1 \smile_{L_1} f_1).$$

Hence

$$d(xy_1 \smile_{L_1} f_1 + f') = x \lim_n d(y_n \smile_{L_n} f_n) = p^j x^2 f.$$

In other words, we have proven:

Proposition 9.20. *Suppose we can pick* $x \in \mathfrak{m}_K$ *and* $y_n \in \mathcal{O}_{L_n}$ *such that* $\mathrm{Tr}_{L_n/K}(y_n) \to x$. *Then*

$$x^2 \cdot H^*(\Gamma, \mathcal{O}_C) = 0$$

in positive degrees. In particular, $H^*(\Gamma, C) = 0$ *in positive degrees.*

So now the game is to prove we can choose y_n as in the proposition. Note that the valuation of the trace increases as we pass to a bigger extension, so there is in general no reason we should expect this to work. Nevertheless, Tate's observation was that if $K/\mathbb{Q}_p$ is 'sufficiently' ramified, we can choose such a sequence; in fact, fixing *any* $x \in \mathfrak{m}_K$, we'll see next week that we can pick y_n as above.

Remark 9.21. Recall that L/K is said to be ramified if

$$e_{L/K} \coloneqq v_L(\pi_K) > 1$$

where v_L is the valuation in L and π_K is a uniformiser of K. In other words, π_L^e divides $\pi_K \in \mathcal{O}_L$ for some $e > 1$.

Ramification is measured by several different gadgets, including the *inverse different*

$$\mathcal{D}_{L/K}^{-1} = \{x \in K \mid \forall y \in \mathcal{O}_L, \mathrm{Tr}_{L/K}(xy) \in \mathcal{O}_K\}.$$

This is an invertible ideal, i.e. an $\mathcal{O}_L$-submodule of L such that there exists another such ideal $\mathcal{D}$ with $\mathcal{D} \cdot \mathcal{D}^{-1} = \mathcal{O}_L$ as submodules of L. The basic connection to our problem is through the following lemma:

Lemma 9.22 ([Ser79], Lemma V.3.4)**.** *If* L/K *is a cyclic extension of prime order* ℓ, *then*

$$\mathrm{Tr}_{L/K}(\pi_L^n \mathcal{O}_L) = \pi_K^r \mathcal{O}_K, \quad \text{where} \quad r = \left\lfloor \frac{n + v_L(\mathcal{D}_{L/K})}{\ell} \right\rfloor.$$

Remark 9.23. If the degree of a cyclic extension L/K is coprime to p then it turns out that $v_L(\mathcal{D}) < \ell$: for example, $\mathrm{Tr}_{L/K}(1) = \ell$ has valuation 0. The only potential issue arises when $\ell = p$.

In particular, if we can bound $v_L(\mathcal{D}_{L/K})$ then we can control the image of the trace. The key point is that as we replace K by increasingly ramified extensions K_N (and L by $L_N \coloneqq LK_N$), the term $v_{L_N}(\mathcal{D}_{L_N/K_N})$ will shrink. We discuss this next week, and combining with Proposition 9.20 deduce:

Theorem 9.24 (Tate)**.** *In the notation as above, one has*

$$H^*(\Gamma_\infty; \mathcal{O}_C) = \begin{cases} \mathcal{O}_{\widehat{K}_\infty}, & * = 0 \\ \textit{almost zero}, & * > 0. \end{cases}$$

Corollary 9.25. *Descending from* K_∞ *to* K *one obtains an almost isomorphism*

$$H^*(\Gamma; \mathcal{O}_C) \cong^a H^*(\mathbb{Z}_p; \mathcal{O}_{\widehat{K}_\infty}).$$

Similar considerations of ramification will allow us also to compute the latter group.

10 Galois cohomology of $\mathcal{O}_C$ (cont.) (12 Aug)

Let's recall some notation from last time. Write $C = \widehat{\overline{\mathbb{Q}}}_p$ for the completion of the algebraic closure, and $\Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for the Galois group; our aim is to compute the groups

$$H^*(\Gamma_{\mathbb{Q}_p}, C(i)), \tag{10.1}$$

at least rationally. The strategy (formulated more generally for a finite extension $K/\mathbb{Q}_p$) was to use an intermediate extension

$$K \xrightarrow{\mathbb{Z}_p} K_\infty \xrightarrow{\Gamma_\infty} C,$$

chosen judiciously so that

1. $H^*(\Gamma_\infty, C) = \widehat{K}_\infty$ is the completion of K_∞ (concentrated in degree zero),
2. $H^*(\mathbb{Z}_p, \widehat{K}_\infty) = H^*(\widehat{K}_\infty \xrightarrow{\sigma-1} \widehat{K}_\infty)$ is computable.

Last time, we discussed the following theorem:

Theorem 10.2 (Ax (H^0), Tate). *If $K_\infty/\mathbb{Q}_p$ is a ramified $\mathbb{Z}_p$-extension and $\Gamma_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$, then*

$$H^*(\Gamma_\infty, \mathcal{O}_C) = \begin{cases} \mathcal{O}_{\widehat{K}_\infty} & * = 0 \\ \textit{almost zero} & * > 0 \end{cases}$$

In particular, the rational cohomology is just $\widehat{K}_\infty$. This theorem relied on the following key result, which we took for granted last time:

Proposition 10.3. *Let L_∞/K_∞ be any finite Galois extension, and $x \in \mathfrak{m}_{K_\infty}$. Then there exists $y \in L_\infty$ with $\mathrm{Tr}_{L_\infty/K_\infty}(y) = x$.*

We also stated the following theorem:

Theorem 10.4. *The inclusion $K \to \widehat{K}_\infty$ induces an equivalence*

$$\mathbf{R}\Gamma(\mathbb{Z}_p, K) = (K \xrightarrow{0} K) \xrightarrow{\sim} \mathbf{R}\Gamma(\mathbb{Z}_p, \widehat{K}_\infty).$$

Using the Lyndon-Hochschild-Serre spectral sequence, Theorems 10.2 and 10.4 combine immediately to give:

Theorem 10.5 (Tate [Tat67]). *There is an equivalence*

$$\mathbf{R}\Gamma(\Gamma_K, C) \simeq K[\varepsilon] := (K \xrightarrow{0} K).$$

Today, we'll complete the proofs of Proposition 10.3 and Theorem 10.4; to make life a bit easier we do this specifically for $K = \mathbb{Q}_p(\zeta_p)$, with K_∞ being the cyclotomic extension. Things are a bit more explicit in that case, and this suffices to compute (10.1), by passing to μ_{p-1}-fixed points. The general case require a little more bookkeeping and a small amount of class field theory, but the proof outline is the same.

Remark 10.6. When K_∞ is the cyclotomic extension of $\mathbb{Q}_p$, the completion $\widehat{K}_\infty$ is the perfectoid field $\mathbb{Q}_p^{\mathrm{cycl}}$ that we considered in Talk 5. In fact, $\widehat{K}_\infty$ is perfectoid for any choice of totally ramified $\mathbb{Z}_p$-extension: indeed, it is a complete nonarchimedean field, and its valuation is nondiscrete since

$$v(\pi_{K_n}) = p^{-p^n} v(\pi_K) \to 0.$$

As such, this computation is the archetypical example of the technique of descent from perfectoid coverings: we can view Proposition 10.3 as a special case of Theorem 5.23.

10.1 Galois theory of ramified extensions

In this section we recall some Galois theory, mostly following [Ser79]. Throughout, we fix a complete discrete valuation field K and a finite Galois extension L/K. We'll assume that the residue fields ℓ/k are both perfect.

Definition 10.7. Write $e_{L/K} := v_L(\pi_K)$, so that $\mathfrak{m}_K \cdot \mathcal{O}_L = \mathfrak{m}_L^{e_{L/K}}$. This is the *ramification index*, and recall that L/K is

- *unramified* if $e_{L/K} = 1$,
- *ramified* if $e_{L/K} > 1$,
- *totally ramified* if $e_{L/K} = [L : K]$.

We have in general an equality $[L : K] = e_{L/K}[\ell : k]$, and so L is totally ramified precisely when the extension on residue fields is trivial. It will also be convenient to write

- $e_K := e_{K/\mathbb{Q}_p}$, the *absolute ramification index*,
- $v := \frac{1}{e_K} v_K \colon K \to \mathbb{Q}$ for the *absolute valuation*. By definition, this extends the valuation on $\mathbb{Q}_p$. Note that
  $$v(x) \geq i \quad \Leftrightarrow \quad |x| \leq |p|^i$$
  where $|-|$ denotes the extension of the norm from $\mathbb{Q}_p$ to K.

Today's key example is the following:

Example 10.8. Write $\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p$ for the nth cyclotomic extension, which is unramified if and only if $(n, p) = 1$. We will therefore focus on $n = q = p^r$; then $\zeta_q^{p^{r-1}}$ is a primitive pth root of unity, so is a root of the cyclotomic polynomial

$$\Phi_p(X) = X^{p-1} + X^{p-2} + \cdots + 1.$$

In particular, $\pi := \zeta_q - 1$ is a root of

$$(X+1)^{p^{r-1}(p-1)} + (X+1)^{p^{r-1}(p-2)} + \cdots + 1,$$

which is Eisenstein: its reduction mod-p is $((X+1)^{p^r} - 1)/((X+1)^{p^{r-1}} - 1) = X^{p^{r-1}(p-1)}$, and the constant term is clearly p. Since $\mathrm{Gal}(\mathbb{Q}_p(\zeta_q)/\mathbb{Q}_p) \subset (\mathbb{Z}/q)^\times$, we deduce that this is a totally ramified $(\mathbb{Z}/q)^\times$-Galois extension, and that π is a primitive element and uniformiser.

We will need the following gadgets that 'measure' ramification:

10.1.1 $\mathcal{D}_{L/K}$

Recall the *inverse different*, which is the $\mathcal{O}_L$-submodule

$$\mathcal{D}_{L/K}^{-1} := \left\{ x \in L : \mathrm{Tr}_{L/K}(xy) \in \mathcal{O}_L \text{ for all } y \in \mathcal{O}_L \right\} \subset L$$

This is an invertible ideal: we define the *different* to be the $\mathcal{O}_L$-submodule of L characterised by $\mathcal{D}_{L/K} \cdot \mathcal{D}_{L/K}^{-1} = \mathcal{O}_L$.

Note that $\mathcal{D}_{L/K}^{-1}$ is the largest invertible ideal of L with trace contained in $\mathcal{O}_K$. In fact:

Lemma 10.9. *If $\mathfrak{a}$ is an invertible ideal of K and $\mathfrak{b}$ an invertible ideal of L, then*

$$\mathrm{Tr}_{L/K}(\mathfrak{b}) \subset \mathfrak{a} \quad \Leftrightarrow \quad \mathfrak{b} \subset \mathfrak{a} \cdot \mathcal{D}_{L/K}^{-1}.$$

Proof. When $\mathfrak{a} = \mathcal{O}_K$, this is precisely the maximality assertion above. For general $\mathfrak{a}$, we have

$$\begin{aligned} \operatorname{Tr}_{L/K}(\mathfrak{b}) \subset \mathfrak{a} &\Leftrightarrow \mathfrak{a}^{-1} \cdot \operatorname{Tr}_{L/K}(\mathfrak{b}) \subset \mathcal{O}_K \\ &\Leftrightarrow \operatorname{Tr}_{L/K}(\mathfrak{a}^{-1} \cdot \mathfrak{b}) \subset \mathcal{O}_K \\ &\Leftrightarrow \mathfrak{a}^{-1} \cdot \mathfrak{b} \subset \mathcal{D}_{L/K}, \end{aligned}$$

since the trace is K-linear. ☐

Corollary 10.10. *If $\mathfrak{b} \subset L$, then*

$$v_K(\operatorname{Tr}_{L/K}(\mathfrak{b})) = \left\lfloor \frac{v_L(\mathfrak{b}) + v_L(\mathcal{D}_{L/K})}{e_{L/K}} \right\rfloor$$

Proof. We have $\mathfrak{b} = \pi_L^i \mathcal{O}_L$; since the trace is K-linear, $\operatorname{Tr}_{L/K}(\mathfrak{b})$ is also an invertible ideal and hence of the form $\pi_K^j \mathcal{O}_K$ (here $i = v_L(\mathfrak{b})$ and $j = v_K(\operatorname{Tr}_{L/K}(j))$). But by Lemma 10.9,

$$\operatorname{Tr}_{L/K}(\mathfrak{b}') \subset \pi_K^j \mathcal{O}_K \quad \Leftrightarrow \quad \mathfrak{b}' \subset \pi_K^j \mathcal{D}_{L/K}^{-1},$$

so j is maximal with

$$i \geq v_L(\pi_K^j \mathcal{D}_{L/K}^{-1}) = e_{L/K} j - v_L(\mathcal{D}_{L/K}).$$ ☐

The trace defines a perfect pairing $\operatorname{Tr}_{L/K} \colon L \times L \to K$; by definition, this restricts to a perfect pairing

$$\operatorname{Tr}_{L/K} : \mathcal{D}_{L/K}^{-1} \times \mathcal{O}_L \to \mathcal{O}_K$$

exhibiting $\mathcal{D}_{L/K}^{-1}$ as the $\mathcal{O}_K$-linear dual $\mathcal{O}_L^\vee$.

The following basic property will be important, and follows by basic manipulations of Lemma 10.9:

Lemma 10.11 ([Ser79], Proposition III.4.8)**.** *If $L/K'/K$ then $\mathcal{D}_{L/K} = \mathcal{D}_{L/K'} \cdot \mathcal{D}_{K'/K}$.*

10.1.2 Ramification groups

Recall that L is totally ramified if and only if the Galois group $G = \operatorname{Gal}(L/K)$ acts trivially on the residue field ℓ. For an arbitrary L, we define

$$G_0 := \{s \in G : s \text{ fixes } \ell\} \subset G,$$

so that L/K factors as

$$K \xrightarrow{\text{unram.}} L^{G_0} \xrightarrow{\text{tot. ram.}} L.$$

More generally, define the *higher ramification groups*

$$\begin{aligned} G_i :=& \{s \in G : s \text{ acts trivially on } \mathcal{O}_L/\pi_L^{i+1}\} \\ =& \{s \in G : v_L(sx - x) \geq i + 1 \text{ for all } x \in \mathcal{O}_L\} \\ =& \{s \in G : v_L(s\alpha - \alpha) \geq i + 1 \text{ for all } x \in \mathcal{O}_L\}, \end{aligned}$$

where $L = K(\alpha)$. The last formulation makes clear that these define a descending sequence

$$G = G_{-1} \triangleright G_0 \triangleright \cdots \triangleright G_t \triangleright G_{t+1} = 1,$$

where $t = \max_{s \neq 1}\{v_L(s\alpha - \alpha)\}$. If $H \subset G$, note that it follows from definitions that $H_i = G_i \cap G$.

Definition 10.12. Write $i_G(s) = v_L(s\alpha - \alpha)$, so that $i_G(s) = i+1$ if and only if $s \in G_i \setminus G_{i+1}$.

Example 10.13. We return to our favourite example $\mathbb{Q}_p(\zeta_q)/\mathbb{Q}_p$. For $s \in G = (\mathbb{Z}/q)^\times$, write $j = \max\{i : s \equiv_{p^i} 1\}$. We claim that

$$i_G(s) = p^j.$$

Indeed, since $\pi_L = \zeta_q - 1$ primitive we have

$$i_G(s) = v_L(s\pi_L - \pi_L) = v_L(\zeta_q^s - \zeta_q) = v_L(\zeta_q^{s-1} - 1).$$

The definition of j implies that ζ^{s-1} is a p^{r-j}th root of unity, and so $\zeta_q^{s-1} - 1$ is a uniformiser of $K' := \mathbb{Q}_p(\zeta_{p^{r-j}})$. Thus

$$i_G(s) = v_L(\pi_{K'}) = e_{L/K'} = p^j.$$

We will therefore write $G(q)^i := \{s : s \equiv_{p^i} 1\} \subset (\mathbb{Z}/q)^\times$. Then for $j \geq 0$, we have

$$G_{p^{j-1}} = \cdots = G_{p^j - 1} = G(q)^j.$$

In particular, $G_{p^{r-1}} = 1$.

10.1.3 Compatibility of G_i and $\mathcal{D}_{L/K}$

According to Corollary 10.10, in order to control the image of the trace we want to control $\mathcal{D}_{L/K}$. We will do this by relating it to the ramification groups, which we can often compute. Throughout, fix $L = K(\alpha)$ and write f_α for the minimal polynomial of α.

Proposition 10.14. $\mathcal{D}_{L/K} = f'_\alpha(\alpha)\mathcal{O}_L$.

Proof. The key claim is that the elements $\alpha^i / f'_\alpha(\alpha)$ give a basis for $\mathcal{O}_L^\vee$, the dual with respect to $\mathrm{Tr}_{L/K}$. Given this, we have $\mathcal{D}_{L/K}^{-1} = f'_\alpha(\alpha)^{-1}\mathcal{O}_L$, so $\mathcal{D}_{L/K} = f'(\alpha)\mathcal{O}_L$.

To prove the claim, we need to show that

$$\det \mathrm{Tr}_{L/K}(\alpha^i \alpha^j / f'_\alpha(\alpha)) \in \mathcal{O}_K^\times. \tag{10.15}$$

Writing $\alpha_1, \cdots, \alpha_n$ for the Galois conjuagtes of α in a splitting field L', we begin by showing that

$$\frac{1}{f_\alpha(T)} = \sum_i \frac{1}{f'_\alpha(\alpha_i)(T - \alpha_i)}. \tag{10.16}$$

To this end, write $1/f(T) = \sum_i A_i/(T - \alpha_i)$, with $A_i \in L'$ determined by the condition

$$\sum_i A_i \prod_{j \neq i} (T - \alpha_i) = 1.$$

To see that $A_i = 1/f'_\alpha(\alpha_i)$, we write

$$g := \sum_i \prod_{j \neq i} \frac{T - \alpha_j}{\alpha_i - \alpha_j} - 1$$

Then g has degree at most $n - 1$. On the other hand, each α_i is a root of g, so $g = 0$; thus $A_i = \prod_{j \neq i} (\alpha_i - \alpha_j)^{-1} = 1/f'_\alpha(\alpha_i)$. We deduce (10.15) by expanding both sides of (10.16) in $X = 1/T$. Indeed,

$$\frac{1}{f(T)} = X^n(1 + f_1(X))^{-1} = X^n + aX^{n+1} + \cdots.$$

On the other hand,

$$\frac{1}{f'_\alpha(\alpha_i)(T-\alpha_i)} = \frac{X}{f'_\alpha(\alpha_i)}(1-\alpha_i X)^{-1} = \frac{X}{f'_\alpha(\alpha_i)}(1+\alpha_i X + \alpha_i^2 X^2 + \cdots).$$

Thus $\mathrm{Tr}_{L/K}(\alpha^i/f'_\alpha(\alpha)) = 0$ for $i < n-2$, while $\mathrm{Tr}_{L/K}(\alpha^{n-1}/f'_\alpha(\alpha)) = 1$. In particular,

$$\mathrm{Tr}_{L/K}(\alpha^i\alpha^j/f'_\alpha(\alpha)) = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & * \end{bmatrix}$$

has determinant $(-1)^n$. $\square$

Using this, we obtain a formula for $v_L(\mathcal{D}_{L/K})$:

Proposition 10.17. *For any L/K,*

$$v_L(\mathcal{D}_{L/K}) = \sum_{s\neq 1} i_G(s) = \sum_{i=0}^{\infty}(|G_i|-1).$$

Remark 10.18. For the second equality, note that the sum is finite.

Proof. We can compute using Proposition 10.14:

$$\begin{aligned} v_L(\mathcal{D}_{L/K}) = v_L(f'_\alpha(\alpha)) &= v_L(\prod_{s\neq 1}(s\alpha-\alpha)) \\ &= \sum_{s\neq 1} v_L(s\alpha-\alpha) \\ &= \sum_{s\neq 1} i_G(s) \end{aligned}$$

This gives the first equality. On the other hand, writing $G_t \supsetneq G_{t+1} = 1$ we have

$$\begin{aligned} \sum_{s\neq 1} i_G(s) &= \sum_{i=0}^{\infty}(i+1)|G_i \setminus G_{i+1}| \\ &= \sum_{i=0}^{t}(i+1)(|G_i|-|G_{i+1}|) \\ &= \sum_{i=0}^{t}|G_i| - (t+1)|G_{t+1}| \\ &= \sum_{i=0}^{\infty}(|G_i|-1). \end{aligned}$$ $\square$

10.2 Proof of Theorems 10.2 and 10.4

Equipped with Proposition 10.17 we can finally prove the main results, starting with Theorem 10.4. The strategy will be to construct an equivariant splitting $t : \widehat{K}_\infty \to K$; at this point we restrict to $K_\infty = \mathbb{Q}_p(\zeta_{p^{1/\infty}})$, writing $K_n = \mathbb{Q}_p(\zeta_{p^{n+1}})$ and $K = K_0 = \mathbb{Q}_p(\zeta_p)$.

Lemma 10.19. *For any $n \geq -1$ we have*

$$v_{K_n}(\mathcal{D}_{K_n/\mathbb{Q}_p}) = p^n((n+1)(p-1)-1)$$

Proof. We computed the ramification groups of K_n in Example 10.13; note that $|G(q)^0| = |G| = p^n(p-1)$, while for $i \geq 1$

$$|G(q)^i| = p^{n-i} \quad \text{and} \quad \#\{j : G_j = G(q)^i\} = p^i(p-1).$$

Applying Proposition 10.17, we compute

$$v_{K_n}(\mathcal{D}_{K_n/\mathbb{Q}_p}) = \sum_{i=0}^{n} p^n(p-1) - p^n = p^n((p-1)-1). \qquad \square$$

From transitivity of the different (Lemma 10.11), we deduce:

Corollary 10.20. *For any $n \geq 0$,*

$$v_{K_{n-1}}(\mathcal{D}_{K_n/K_{n-1}}) = p^n(p-1).$$

Definition 10.21. Set $t := p^{-n}\operatorname{Tr}_{K_n/K} : K_n \to K$. Note that this is compatible with the inclusions $K_n \hookrightarrow K_{n+1}$, so extends to a section

$$t : K_\infty \to K.$$

Proposition 10.22. *t is continuous, so extends uniquely to a section*

$$t : \widehat{K}_\infty \to K.$$

Proof. Plugging Corollary 10.20 into Corollary 10.10 gives

$$\begin{aligned} v_{K_{n-1}}(\operatorname{Tr}_{K_n/K_{n-1}}(x)) &\geq \left\lfloor \frac{v_{K_n}(x) + p^n(p-1)}{p} \right\rfloor \\ &\geq \frac{v_{K_n}(\operatorname{Tr}_{K_n/K_{n-1}}(x))}{p} + p^{n-1}(p-1) - 1 \end{aligned}$$

for any $x \in K_n$. In terms of the absolute valuation $v = v_{K_{n-1}}/p^n(p-1)$, this reads

$$v(\operatorname{Tr}_{K_n/K_{n-1}}(x)) \geq v(x) + 1 - 1/p^n(p-1),$$

or equivalently

$$|\operatorname{Tr}_{K_n/K_{n-1}}(x)| \leq |p|^{1-p^{-n}(p-1)^{-1}}|x|.$$

Applying this iteratively, we have

$$|\operatorname{Tr}_{K_n/K}(x)| = |\operatorname{Tr}_{K_1/K_0} \cdots \operatorname{Tr}_{K_n/K_{n-1}}(x)| \leq |p|^{n-(p-1)^{-2}}|x|$$

using $\sum_{i=1}^{n} p^{-n} \leq (p-1)^{-1}$. In particular,

$$|t(x)| \leq |p|^{-(p-1)^{-2}}|x|$$

for any $x \in K_\infty$. $\square$

Remark 10.23. The key point is to get a bound on $|t(x)|$ for $x \in K_n$ which is uniform in n; Tate uses essentially the same method to prove that this holds for arbitrary K_∞/K, though the constant is less explicit.

Corollary 10.24. *Writing* $X \coloneqq \ker(t)$*, we have*

$$H^*(\mathbb{Z}_p, \widehat{K}_\infty) = H^*(\mathbb{Z}_p, K) \oplus H^*(\mathbb{Z}_p, X).$$

Pick a topological generator $\sigma \in \mathbb{Z}_p$. To complete the proof of Theorem 10.4, we need to show that

$$\sigma - 1 : X \xrightarrow{\sim} X.$$

Lemma 10.25. *There exists a constant* $b > 0$ *such that for all* $x \in K_\infty$*,*

$$|t(x) - x| \leq b|\sigma x - x|. \tag{10.26}$$

In particular, $\sigma - 1\colon X \to X$ *is injective.*

Proof. We define $t_n \coloneqq p^{-1}\operatorname{Tr}_{K_n/K_{n-1}}\colon K_n \to K_{n-1}$, so that $t = t_1 \circ \cdots \circ t_n$. We begin by inductively finding constants b_n such that (10.26) holds with $b = b_n$ whenever $x \in K_n$. Since $\sigma^{p^{n-1}}$ generates $\mathrm{Gal}(K_n/K_{n-1})$, we can compute

$$\begin{aligned}
|t_n(x) - x| &= \left| p^{-1} \sum_{i=0}^{p-1} \sigma^{p^{n-1}i} x - x \right| \\
&= |p|^{-1} \left| \sum_{i=0}^{p-1} (\sigma^{p^{n-1}i} x - x) \right| \\
&= |p|^{-1} \left| \sum_{i=0}^{p-1} (1 + \sigma^{p^{n-1}} + \cdots + \sigma^{p^{n-1}(p-2)})(\sigma^{p^{n-1}} x - x) \right| \\
&\leq |p|^{-1} \left| \sigma^{p^{n-1}} x - x \right| \\
&\leq |p|^{-1} \left| (1 + \sigma + \cdots + \sigma^{p^{n-1}-1})(\sigma x - x) \right| \\
&\leq |p|^{-1} \left| \sigma x - x \right|
\end{aligned}$$

For $x \in K_1$, this tells us that $|t(x) - x| \leq |p|^{-1}|\sigma x - x|$. For $x \in K_n$ with $n > 2$ we have $t(x) = t(t_n(x))$, so

$$|t(x) - x| = |t(t_n(x)) - t_n(x) + t_n(x) - x| \leq \max\{|t(t_n(x) - t_n(x)|, |t_n(x) - x|\}.$$

But

$$|t(t_n(x)) - t_n(x)| \leq b_{n-1}|\sigma t_n(x) - t_n(x)| \leq b_{n-1}|p|^{-p^n(p-1)^{-1}}|\sigma x - x|$$

by induction and Corollary 10.20, while

$$|t_n(x) - x| \leq |p|^{-1}|\sigma x - x|.$$

Thus we can take $b_n = \max\{|p|^{-p^n(p-1)^{-1}} b_{n-1}, |p|^{-1}\}$. In particular, b_n is uniformly bounded by $|p|^{-1-(p-1)^{-2}}$. ☐

Proof. (Theorem 10.4). We must show that $\sigma - 1$ is an isomorphism on X. By Lemma 10.25 we know it is injective, and in particular is injective when restricted to each $X_n \coloneqq X \cap K_n$. These are finite dimensional K-vector spaces, and so $(\sigma - 1)|_{X_n}$ is also surjective; passing to the colimit we see that $(\sigma - 1)|_{X_\infty}$ is bijective, and pick an inverse $\rho = (\sigma - 1)^{-1}$. Applying (10.26), we have for $x \in X_\infty$

$$|\rho(x)| \leq b|(\sigma - 1)\rho(x)| = b|x|,$$

so ρ extends to an inverse on X. ☐

Remark 10.27. A similar proof shows that the cohomology of nontrivial twists of $\widehat{K}_\infty$ is zero.

This concludes the proof of Theorem 10.4. For Theorem 10.2, we will need one more bookkeeping tool.

10.2.1 Upper indexing

Definition 10.28. Let $\varphi = \varphi_{L/K} \colon [-1, \infty) \to [-1, \infty)$ be the *Herbrand function*, the continuous piecewise linear function determined by $\varphi(-1) = -1$ and

$$\varphi'(v) = \left[G_0 : G_{\lfloor v \rfloor}\right]^{-1}.$$

In particular, at integers we have $\varphi(n) = |G_0|^{-1} \sum_{i=0}^{n} |G_i|$.

Definition 10.29. Since φ is strictly increasing, we can write $\psi \coloneqq \varphi^{-1}$. The *upper indexing* on the ramification groups is defined as

$$G^i \coloneqq G_{\psi(i)}.$$

Remark 10.30. Substituting $v = \psi(u)$, Proposition 10.17 becomes

$$v_L(\mathcal{D}_{L/K}) = \sum_{i=0}^{\infty}(|G_i| - 1) = \int_{-1}^{\infty}(|G_v| - 1)dv = |G_0|^{-1} \int_{-1}^{\infty} 1 - |G^u|^{-1} du$$

with respect to the upper indexing.

Example 10.31. For $L = \mathbb{Q}_p(\zeta_q)/\mathbb{Q}_p$, the Herbrand function is piecewise linear with gradient jumps at (p^i, i). Thus $G^i = G(q)^i$ for all i.

Recall that the groups G_i are compatible with restriction to subgroups. The key property of the upper indexing is:

Proposition 10.32 ([Ser79], Proposition IV.3.14)**.** *Let $H \triangleleft G$ be a normal subgroup. Then for $i \geq -1$,*

$$(G/H)^i = G^i H/H.$$

This is a straightforward consequence of the computation

$$i_{G/H}(s') = \frac{1}{e_{L/K'}} \sum_{\bar{s} = s'} i_G(s),$$

where L/K' corresponds to H. This is [Ser79, Proposition IV.1.3], which Serre attributes also to Tate.

10.2.2 Proof of Theorem 10.2

Finally, we come to the proof of Theorem 10.2. Recall that we had the following left to prove:

Proposition 10.33 (Proposition 10.3)**.** *Let L_∞/K_∞ be a finite Galois extension, and $x \in \mathcal{O}_{K_\infty}$ with $v(x) > 0$. Then there exists $y \in \mathcal{O}_{L_\infty}$ with $\mathrm{Tr}_{L_\infty/K_\infty}(y) = x$.*

Proof. We can find $N \gg 0$ and a finite Galois extension L_N/K_N such that $L_\infty = K_\infty \cdot L_N$: if $L_\infty = K_\infty(\alpha)$, then we just need f_α to be defined over K_n. Increasing N if necessary, we can assume that $x \in K_N$. We must therefore show that

$$x \in \mathrm{Tr}_{L_n/K_n}(\mathcal{O}_{L_n})$$

for $n \gg N$, which we'll do using Proposition 10.17. Indeed, by transitivity we have

$$\begin{aligned} v(\mathcal{D}_{L_n/K_n}) &= v(\mathcal{D}_{L_n/K}) - v(\mathcal{D}_{K_n/K}) \\ &= \int_{-1}^{\infty} |\mathrm{Gal}(K_n/K)^u|^{-1} - |\mathrm{Gal}(L_n/K)^u|^{-1} du \end{aligned}$$

Claim: There exists m (independent of n) such that

$$\mathrm{Gal}(K_n/K)^t = \mathrm{Gal}(L_n/K)^t$$

for $t \geq m$.

In particular, this implies that

$$\begin{aligned} v(\mathcal{D}_{L_n/K_n}) &= \int_{-1}^{m} |\mathrm{Gal}(K_n/K)^u|^{-1} - |\mathrm{Gal}(L_n/K)^u|^{-1} du \\ &\leq \int_{-1}^{m} |\mathrm{Gal}(K_n/K)^u|^{-1} du \\ &= \sum_{i=0}^{m} p^{i-n} = p^{-n} \frac{p^{m+1}-1}{p-1} \xrightarrow{n\to\infty} 0, \end{aligned}$$

so $\mathrm{Tr}_{L_n/K_n}(\mathcal{O}_{L_n}) \ni x$ for $n \gg 0$, by Corollary 10.10. To prove the claim note that

$$\mathrm{Gal}(K_n/K)^t = (\mathrm{Gal}(L_n/K)/\mathrm{Gal}(L_n/K_n))^t = \mathrm{Gal}(L_n/K)^t \mathrm{Gal}(L_n/K_n)/\mathrm{Gal}(L_n/K_n),$$

so it suffices to find m such that $\mathrm{Gal}(L_n/K)^m \cap \mathrm{Gal}(L_n/K_n) = 1$; to find m that works uniformly, we are looking for m such that

$$\mathrm{Gal}(L_\infty/K)^m \cap \mathrm{Gal}(L_\infty/K_\infty) = 1.$$

Such an m exists, since $\mathrm{Gal}(L_\infty/K_\infty)$ is finite and $\mathrm{Gal}(L_\infty/K)^t$ converge to 1. This concludes the proof of Proposition 10.3, and hence of Theorem 10.2. ☐

11 Construction of approximation maps (15 Aug)

11.1 Review of the general strategy

Let's recall the setup and approach for [Bar+24, Theorem B] (as in Talk 9). We fix a prime p, formal group Γ of height h over $\overline{\mathbb{F}}_p$, LT $\cong$ Spf A the Lubin-Tate deformation space of Γ, and $\mathcal{O}_D = \mathrm{End}\,\Gamma$; this fits in a short exact sequence

$$1 \to \mathcal{O}_D^\times \to \mathbb{G}_h \to \mathrm{Gal}(\mathbb{F}_p) \to 1,$$

in which the central term is the group of automorphisms of the pair $(\overline{\mathbb{F}}_p, \Gamma)$. Our goal is to compute the continuous cohomology

$$H^s(\mathbb{G}_h; A[u_h^{\pm 1}]) = E_2^{s,t},$$

at least after rationalisation. The strategy is to pass to the generic fibre LT_K (where $K \colon= W(\overline{\mathbb{F}}_p)[1/p]$), which fits in a span

$$\begin{tikzcd} & \mathfrak{X} \arrow[dl, "\mathrm{GL}_h(\mathbb{Z}_p)"'] \arrow[dr, "\mathbb{G}_h"] & \\ \mathrm{LT}_K & & \mathcal{H} \end{tikzcd}$$

whose legs are proétale torsors of stacks. As a result, we obtain the equivalences

$$\mathbf{R}\Gamma(\mathrm{LT}_{K,\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})^{h\mathbb{G}_h} \simeq \mathbf{R}\Gamma(\mathfrak{X}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})^{h(\mathbb{G}_h \times \mathrm{GL}_h(\mathbb{Z}_p))} \simeq \mathbf{R}\Gamma(\mathcal{H}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})^{h\mathrm{GL}_h(\mathbb{Z}_p)} \quad (11.1)$$

in $\mathcal{D}(\mathrm{Solid})$. The left-hand side is supposed to be related to $H^*(\mathbb{G}_h, A)$, and the right-hand to (more computable) $\mathrm{GL}_h(\mathbb{Z}_p)$-cohomology. Today's goal is to make this precise:

Theorem 11.2 (Approximation theorem, [Bar+24] Theorem 3.9.3).

1. *There is a $\mathbb{G}_h$-equivariant morphism of DG solid W-algebras*
$$\alpha_{\mathrm{LT}_K} : A[\varepsilon] \to \mathbf{R}\Gamma(\mathrm{LT}_{K,\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}}),$$
where $A[\varepsilon] \coloneqq (A \xrightarrow{0} A)$ in degrees 0 and 1.
2. *There is a $\mathrm{GL}_h(\mathbb{Z}_p)$-equivariant morphism of DG solid $\mathbb{Z}_p$-algebras*
$$\alpha_{\mathcal{H}} : \mathbb{Z}_p[\varepsilon] \to \mathbf{R}\Gamma(\mathcal{H}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}}).$$
3. *Write $C_{\mathcal{Y}} \coloneqq \mathrm{cofib}(\alpha_{\mathcal{Y}}) \in \mathcal{D}(\mathrm{Solid})$, where $\mathcal{Y} = \mathrm{LT}_K$ or $\mathcal{H}$. Then there exists some $N > 0$ such that*
$$H^i(C_{\mathcal{Y}}) = \begin{cases} 0 & i \leq 0 \\ p^N\text{-torsion} & i \geq 1 \end{cases}$$

Example 11.3. When $\mathcal{Y} = \operatorname{Spa} K$, we saw in Emma's talk that proétale cohomology of $\mathcal{Y}$ agrees with continuous group cohomology, at least when the coefficients are solid. The analogous statement is therefore (the integral version of) Tate's theorem Theorem 10.5, which we discussed last week.

Combining (11.1) and Theorem 11.2 gives the following sequence of maps

$$\begin{aligned} A^{h\mathbb{G}_h} \otimes \mathbb{Z}_p[\varepsilon] \simeq A[\varepsilon]^{h\mathbb{G}_h} &\xrightarrow{\alpha} \mathbf{R}\Gamma(\mathrm{LT}_{K,\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})^{h\mathbb{G}_h} \\ &\simeq \mathbf{R}\Gamma(\mathcal{H}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})^{h\mathrm{GL}_h(\mathbb{Z}_p)} \\ &\xleftarrow{\alpha} \mathbb{Z}_p[\varepsilon]^{h\mathrm{GL}_h(\mathbb{Z}_p)} \simeq \mathbb{Z}_p^{h\mathrm{GL}_h(\mathbb{Z}_p)} \otimes [\varepsilon]. \end{aligned}$$

whose cofibre is p-power torsion. In particular, after inverting p we get

$$H^*(\mathbb{G}_h; A[1/p]) \otimes \mathbb{Q}_p[\varepsilon] \cong H^*(\mathrm{GL}_h(\mathbb{Z}_p); \mathbb{Q}_p) \otimes \mathbb{Q}_p[\varepsilon].$$

The $\mathbb{G}_h$-splitting $A \cong W \oplus A^c$ in Proposition 1.40, together with Lazard's computation of rational cohomology of p-adic Lie groups, allows us to conclude that

$$H^*(\mathbb{G}_h, A^c[1/p]) = 0,$$

by counting dimensions on each side.

11.1.1 The general comparison theorem

In fact, Theorem 11.2 is a special case of a more general theorem in p-adic geometry:

Theorem 11.4 ([Bar+24], Corollary 5.6.9)**.** *Let $\mathfrak{X}$ be a quasi-separated* semistable *formal scheme over $\mathcal{O}_K$ with generic fibre X. There is a natural morphism in $\mathcal{D}(\mathrm{Cond})$,*

$$\alpha_{\mathfrak{X}} : \mathbf{R}\Gamma(\mathfrak{X}, \mathcal{O}_{\mathrm{cond}})[\varepsilon] \to \mathbf{R}\Gamma(X_{\mathrm{proét}}, \widehat{\mathcal{O}}^{+}_{\mathrm{cond}}),$$

whose cofibre $C_{\mathfrak{X}}$ has p-power torsion cohomology.

We will discuss the precise meaning of *semistability* in Talk 11.3. For now, note that passing to global sections gives:

Corollary 11.5 ([Bar+24], Theorem D)**.** *Let K be a local field of characteristic $(0, p)$, and $\mathfrak{X}, X$ as above. There is a natural isomorphism of graded K-vector spaces*

$$H^*(\mathfrak{X}, \mathcal{O}) \otimes_{\mathcal{O}_K} K[\varepsilon] \cong H^*(X_{\mathrm{proét}}, \widehat{\mathcal{O}}^{+}_{\mathrm{cond}}) \otimes_{\mathcal{O}_K} K,$$

where the right-hand side denotes coherent cohomology, and $|\varepsilon| = 1$.

The general philosophy is to use the maps of sites[13]

$$X_{\mathrm{proét}} \to X_{\mathrm{ét}} \to X_{\mathrm{an}},$$

which exists for any rigid analytic X over K and gives rise to comparisons between the cohomology of the structure sheaf in each case. Recall Talk 4 that when $X = \operatorname{Spa} K$, we have

- $H^*(X_{\mathrm{an}}, \mathcal{O}_X) = K$;
- $H^*(X_{\mathrm{ét}}, \mathcal{O}_X) = H^*(\mathrm{Gal}(\overline{K}/K), \overline{K})$ is Galois cohomolgy of the algebraic closure equipped with the *discrete* topology, which is K by the normal basis theorem;
- $H^*(X_{\mathrm{proét}}, \widehat{\mathcal{O}}_X) = H^*(\mathrm{Gal}(\overline{K}/K), C)$ is Galois cohomology of the completed algebraic closure $C = \widehat{\overline{K}}$, which is $K[\varepsilon]$ by Theorem 10.5.

In fact, by étale descent it follows that

$$H^*(X_{\mathrm{an}}, \mathcal{O}_X) \cong H^*(X_{\mathrm{ét}}, \mathcal{O}_X)$$

for any smooth rigid analytic X over K. On the other hand, $H^*(X_{\mathrm{proét}}, \mathcal{O}_X)$ might look very different.

11.2 The rational case

In this section, suppose that C is as above and that X is a smooth rigid-analytic space over C. Write $\nu\colon X_{\mathrm{proét}} \to X_{\mathrm{ét}}$ for the morphism of sites. The key input from p-adic geometry is the following striking computation:

Theorem 11.6 (Scholze, [Sch13b, Proposition 3.23])**.** *There is an isomorphism of $\mathcal{O}_{X_{\mathrm{ét}}}$-modules*

$$\Omega^j_{X_{\mathrm{ét}}}(-j) \cong \mathbf{R}^j \nu_* \widehat{\mathcal{O}}_X$$

between the (Tate-twisted) module of differential j-forms, and the jth derived pushforward of the proétale structure sheaf.

---

[13] Following [Bar+24], this means there are functors going in the opposite directions

Sketch. Since X is smooth, there is locally on $X_{\text{ét}}$ an étale map to a rigid-analytic torus

$$\mathbb{T}^d := \operatorname{Spa}(R_d, R_d^+), \quad R_d^+ := \mathcal{O}_C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle.$$

In the case of a torus, we use the perfectoid cover $\widetilde{\mathbb{T}}^d \to \mathbb{T}^d$, where

$$\widetilde{\mathbb{T}}^d := \operatorname{Spa}(\widetilde{R}_d, \widetilde{R}_d^+), \quad \widetilde{R}_d^+ := \mathcal{O}_C\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle,$$

which is a $\mathbb{Z}_p(1)^d$-torsor (and we write $\mathbb{Z}_p(1) = \lim \mu_{p^r}$). We know that affinoid perfectoids are acyclic, so get the first isomorphism in

$$H^i(\mathbb{T}^d, \widehat{\mathcal{O}}_{\mathbb{T}^d}) \cong H^i(\mathbb{Z}_p(1)^d, \widetilde{R}_d) \cong H^i(\mathbb{Z}_p(1)^d, R_d) \cong \Lambda^i_{R_d} H^1(\mathbb{Z}_p(1)^d, R_d).$$

The last isomorphism is Künneth, and the middle [Sch13a, Proposition 5.5]. But $H^1(\mathbb{Z}_p(1)^d, R_d)$ is a free R_d-module of rank d; using this, together with the map $d\log\colon \mathcal{O}_X^\times \to \Omega_X^1$ and the exact sequence

$$0 \to \mathbb{Z}_p(1) \longrightarrow \lim_{\times p} \mathcal{O}_X^\times \longrightarrow \mathcal{O}_X^\times \to 1,$$

Scholze constructs in [Sch13b, Proposition 3.24] an isomorphism $\Omega^1_{X_{\text{ét}}}(1) \to \mathbf{R}^1\Gamma(X_{\text{ét}}, \nu_*\widehat{\mathcal{O}}_X)$. □

IM:Could we add this ‘direct computation’?

Next we suppose that X is rigid-analytic over K, and descend from $\overline{X} := X \times_K C$.

Proof. (Corollary 11.5, rational case). Write $\Gamma_K := \operatorname{Gal}(\overline{K}/K)$, and $\overline{\nu} : \overline{X}_{\text{proét}} \to \overline{X}_{\text{ét}}$. Then

$$\mathbf{R}\nu_*\widehat{\mathcal{O}}_X \simeq \mathbf{R}\overline{\nu}_*(\widehat{\mathcal{O}}_{\overline{X}}^{h\Gamma_K}) \simeq (\mathbf{R}\overline{\nu}_*\widehat{\mathcal{O}}_{\overline{X}})^{h\Gamma_K},$$

and by Theorem 11.6 this admits a filtration with graded part

$$\operatorname{gr}_j \mathbf{R}\nu_*\widehat{\mathcal{O}}_X = \Omega^j_{\overline{X}_{\text{ét}}}(-j)^{h\Gamma_K}.$$

If $U \subset X$ is an affinoid open, then

$$\begin{aligned} \mathbf{R}\Gamma(U, \Omega^j_{\overline{X}_{\text{ét}}}(-j)^{h\Gamma_K}) &\simeq \mathbf{R}\Gamma(U, \Omega^j_{\mathcal{O}(U)/K} \otimes_K C(-j))^{h\Gamma_K} \\ &\simeq \Big[\mathbf{R}\Gamma(U, \Omega^j_{\mathcal{O}(U)/K}) \otimes_K C(-j)\Big]^{h\Gamma_K} \\ &\simeq \mathbf{R}\Gamma(U, \Omega^j_{\mathcal{O}(U)/K}) \otimes_K C(-j)^{h\Gamma_K}. \end{aligned}$$

IM: Justify these

But now we invoke Theorem 10.5 to conclude that

$$\Gamma(U, \mathbf{R}^j\nu_*\widehat{\mathcal{O}}_{X_{\text{ét}}}) = \begin{cases} \mathcal{O}(U)[\varepsilon] & j = 0 \\ 0 & j \neq 0 \end{cases}$$

That is, $\mathbf{R}\nu_*\widehat{\mathcal{O}}_X \simeq \mathcal{O}_{X_{\text{ét}}}[\varepsilon]$. Taking global sections gives the result. □

11.3 The integral and affine case

In this section, we’ll sketch the necessary modifications for the integral version of Corollary 11.5. The first thing we need is an integral version of Theorem 11.6, which is a much more difficult result. When X has a sufficiently nice formal model over $\mathcal{O}_C$, results of [BMS18; ČK19] allow us to control the cohomology of $\overline{\nu}_*\widehat{\mathcal{O}}_{\overline{X}}$. While not used in the proof of the rational vanishing conjecture, this allows in principle to bound the p-power torsion in the integral comparison.

Definition 11.7. Let $\mathfrak{X}$ be a formal scheme over $\mathcal{O}_K$. We say that $\mathfrak{X}$ is *semistable* of dimension d if it is covered by affine opens $\mathfrak{U}$ which admit an étale $\mathcal{O}_K$-morphism to

$$\operatorname{Spf} \mathcal{O}_K\langle T_0, \ldots, T_r, T_{r+1}^{\pm 1}, \ldots, T_d^{\pm 1}\rangle/(T_0 \cdots T_r - \pi), \tag{11.8}$$

where $\pi \in \mathcal{O}_K$ has positive valuation.

Remark 11.9. In the definition, we do not require r or π to be fixed (over $\mathfrak{X}$). The idea is that we allow some mild singularities in $\mathfrak{X}$.

The definition of semistable allows us to replace $\Omega^1_{\mathfrak{X}_{\text{ét}}}$ by the sheaf of *log-differentials*

$$\Omega^1_{\mathfrak{X}_{\text{ét}},\log} := \Omega^1_{\mathfrak{X}_{\text{ét}}} \oplus \mathcal{O}_{\mathfrak{X}_{\text{ét}}}\{df/f : f \in \mathcal{O}_{\mathfrak{X}_{\text{ét}}} \cap \mathcal{O}_{\mathfrak{X}}[1/p]^\times\},$$

which is a locally free $\mathcal{O}_{\mathfrak{X}}$-module of rank d; that is, $\mathfrak{X}$ is *log-smooth*. Set also

$$\Omega^j_{\mathfrak{X}_{\text{ét}},\log} := \Lambda^j \Omega^1_{\mathfrak{X}_{\text{ét}},\log}.$$

If $\mathfrak{X}$ is affine semistable then its rigid-analytic generic fibre $X = \mathfrak{X}_K$ is a smooth affinoid. Moreover, in this case we have:

Lemma 11.10. *The canonical map*

$$\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \to \widehat{\mathcal{O}}^+_X(X)$$

is an equivalence.

Proof. We can restrict to the case in which $\mathfrak{X}$ is given by (11.8), since it is enough to check étale locally on $\mathfrak{X}$. Write $X = \operatorname{Spf} R$, and recall that $X = \mathfrak{X} \times_{\operatorname{Spf} \mathcal{O}_K} \operatorname{Spa} K$. To compute this pullback of adic spaces, note that the structure map $\mathfrak{X} \to \operatorname{Spf} \mathcal{O}_K$ is adic, since the topology on R is π-adic and so any open ideal has open intersection with $\mathcal{O}_K$. Thus

$$\mathcal{O}^+_X(X) = D^+,$$

the integral closure of R in $R[1/\pi]$. But $R \subset R[q/\pi]$ is integrally closed: in fact it is a DVR, which is stronger. ☐

Next, note that if $\mathfrak{X}$ is a formal scheme, we have a morphism of sites $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{ét}}$ sending $\mathfrak{U}$ étale over $\mathfrak{X}$ to $U = \mathfrak{U}_K$. This gives an integral comparison map

$$\mathcal{O}_{\mathfrak{X}} \to \mathbf{R}\nu_* \widehat{\mathcal{O}}^+_X$$

on $X_{\text{ét}}$. To formulate the general comparison theorem in the integral we need two more ingredients from [BMS18]:

IM: Add some more detail on these

1. The *Breuil-Kisin-Fargues twist*

   $$\mathcal{O}_C\{1\} := \widetilde{\xi} A_{\text{inf}}/\widetilde{\xi}^2 A_{\text{inf}}$$

   is a free $\mathcal{O}_C$-module of rank one, with twisted Frobenius action; here $\widetilde{\xi}$ generates the kernel of $A_{\text{inf}} \to \mathcal{O}_C$. This contains $\mathcal{O}_C(1)$ as a submodule, with the cokernel having annihilator $(\zeta_p - 1)$. For M a φ-semilinear $\mathcal{O}_C$-module and $j \in \mathbb{Z}$, write $M\{j\} := M \otimes_{\mathcal{O}_C} \mathcal{O}_C\{1\}^{\otimes j}$.
2. The *décalage functor* $L\eta = L\eta_{\mathcal{I}} \colon \mathcal{D}(\mathfrak{X}_{\text{ét}}) \to \mathcal{D}(\mathfrak{X}_{\text{ét}})$, defined for any invertible subsheaf $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$, has the effect of killing f-torsion in cohomology. It is lax symmetric monoidal and preserves filtered colimits and truncations, but is *not* exact.

With this in place, we can state the analogue of Theorem 11.6 in the integral case:

Theorem 11.11 ([BMS18], Theorem 8.3)**.** *Let $\mathfrak{X}$ be a semistable affine formal scheme over $\mathcal{O}_C$. For all $j \geq 0$ there is a natural isomorphism of $\mathcal{O}_{\mathfrak{X}}$-modules*

$$\Omega^j_{\mathfrak{X},\log}\{-j\} \cong H^j\left(L\eta_{(\zeta_p-1)}\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X\right).$$

Definition 11.12. In light of the theorem, write $\widetilde{\Omega}_{\mathfrak{X}} := L\eta_{(\zeta_p-1)}\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X \in \mathcal{D}(\mathfrak{X}_{\text{ét}})$.

For arbitrary base field K, we can apply this to $\overline{\mathfrak{X}} := \mathfrak{X} \times_{\mathcal{O}_K} \mathcal{O}_C$, using $\Omega^j_{\overline{\mathfrak{X}},\log} \cong \Omega^j_{\mathfrak{X},\log}\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_C$ to deduce:

Corollary 11.13. *Let $\mathfrak{X}$ be a semistable affine formal scheme over $\mathcal{O}_K$. For all $j \geq 0$ there is a Γ_K-equivariant isomorphism of $\mathcal{O}_{\mathfrak{X}}$-modules*

$$\Omega^j_{\mathfrak{X},\log}\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_C\{-j\} \cong H^j(\widetilde{\Omega}_{\overline{\mathfrak{X}}}).$$

To define $\alpha_{\mathfrak{X}}$ for affine semistable $\mathfrak{X}$, we begin with the natural transformation

$$\nu^* : \mathcal{O}_{\mathfrak{X}} \to \nu_*\widehat{\mathcal{O}}^+_X,$$

adjoint to the map which pulls back functions on $\mathfrak{X}$ to integral functions on X. Since $\mathcal{O}_{\mathfrak{X}}$ is acyclic, this gives on global sections a map of rings

$$\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \to \mathbf{R}\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}^+_X).$$

On the other hand, we have the structure morphism $s\colon X \to \operatorname{Spa} K$, which induces a map

$$\mathcal{O}_K[\varepsilon] \xrightarrow{\alpha_{\operatorname{Spf}\mathcal{O}_K}} \mathcal{O}_C^{h\Gamma_K} \simeq \mathbf{R}\Gamma(\operatorname{Spa} K_{\text{proét}}, \mathcal{O}_{\operatorname{Spa} K}) \xrightarrow{s^*} \mathbf{R}\Gamma(X_{\text{proét}}, \mathcal{O}_X)$$

of algebras in $\mathcal{D}(\text{Solid})$; here $\alpha_{\operatorname{Spf}\mathfrak{O}_K}$ is induced by $\mathcal{O}_K \to \mathcal{O}_C$. Note that

$$H^i(\operatorname{cofib}(\alpha_{\operatorname{Spf}\mathcal{O}_K})) = \begin{cases} 0 & i = 0 \\ p^{M_K}\text{-torsion} & i = 1 \\ p^2\text{-torsion} & i \geq 2 \end{cases}$$

by the refinement of Tate's theorem [Bar+24, Theorem C], where M_K is some explicit constant depending only on K.

Altogether, we get the map

$$\alpha_{\mathfrak{X}} : \mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \otimes_{\mathcal{O}_K} \mathcal{O}_K[\varepsilon] \to \mathbf{R}\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}^+_X).$$

Finally, here is the form of Corollary 11.5 proven in the affine case:

Theorem 11.14. *Let $\mathfrak{X}$ be an affine semistable formal scheme of dimension d over K. Then for all $i \geq 0$, $H^i(\operatorname{cofib}(\alpha_{\mathfrak{X}}))$ is killed by $p^{M_K(d)}$. Here $M_K(d)$ is an explicit linear function of d depending only on K; in fact, $M_K(d)$ is insensitive to tame Galois extensions of K.*

Sketch. The strategy is essentially as in the rational case, but we have to work harder: again we descend from $\mathcal{C}$. To begin with, [Bar+24, Lemma 5.5.5] bounds the p-torsion in

$$H^i\left(\operatorname{cofib}(\widetilde{\Omega}_{\overline{\mathfrak{X}}} \to \mathbf{R}\overline{\nu}_*\widehat{\mathcal{O}}^+_{\overline{X}})\right),$$

using some basic properties of $L\eta$ proven in [BMS18, §6]. On the other hand, Corollary 11.13 bounds the torsion in

$$H^i\left(\operatorname{cofib}(\mathcal{O}_{\overline{\mathfrak{X}}}(\overline{\mathfrak{X}}) \to \widetilde{\Omega}_{\overline{\mathfrak{X}}})\right),$$

and hence of $H^i(\operatorname{cofib}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \to \widetilde{\Omega}_{\mathfrak{X}}))$ Combining these bounds gives the theorem. □

11.4 The general case

Finally, we consider the case that X is an arbitrary smooth rigid analytic space over K. Then X need not admit a semistable formal model over $\mathcal{O}_X$, but there is the following result:

Theorem 11.15 ([Har03], Theorem 1.4)**.** *$X_{\text{ét}}$ admits an open cover by rigid-analytic subspaces U with each $U = \mathfrak{U}_L$ for an affine semistable formal scheme over $\mathcal{O}_L$ and L/K finite.*

On each U we can apply Theorem 11.14. However, when X is not quasicompact (e.g. $X = \mathrm{LT}_K$) we could in theory have fields L of increasing ramification index, which could introduce accumulating constants when we descend to X. This problem goes away if we assume that the extensions L/K in Theorem 11.15 are all tamely ramified.

Lemma 11.16. *Let X be a rigid analytic space over K and define*

$$\beta_X : \mathcal{O}_X^+(X)[\varepsilon] \to \mathbf{R}\Gamma(X_{\text{ét}}, \mathcal{O}^+)[\varepsilon] \xrightarrow{\nu_X} \mathbf{R}\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}^+)$$

Assume there is a finite tame Galois extension L/K such that $X = \mathfrak{X}_L$, with $\mathfrak{X}$ affine semistable over $\mathcal{O}_L$. Then there exists a constant N_K depending only on K such that $H^i(\mathrm{cofib}(\beta_X))$ is p^{N_K}-torsion.

Proof. Combine Lemma 11.10 and Theorem 11.14. □

IM: Spell this out

Using this we will pass to the non-affine case.

Definition 11.17. Let X/K be a smooth rigid analytic space of dimension d and $\mathcal{U} = \{U_i\}_I$ an open cover of X_{an}. We say that $\mathcal{U}$ is *tamely semistable* if for each $J \subset I$, there exists a tamely ramified Galois extension L_J/K and a semistable affine formal scheme $\mathfrak{U}_J/\mathcal{O}_{L_J}$ such that

$$U_J := \bigcap_{i\in J} U_i \cong (\mathfrak{U}_{J,L_J}).$$

Such a $\mathcal{U}$ automatically exists when X itself admits a semistable formal model over $\mathcal{O}_K$ (even if this is not affine).

Definition 11.18. Given a covering $\mathcal{U}$ of a rigid analytic space, define the *condensed Čech complex* to be

$$\check{C}(\mathcal{U}; \mathcal{O}^+_{\mathrm{cond}}) := \lim_{[n]\in\Delta} \prod_{f\colon [n]\to I} \mathcal{O}^+_{\mathrm{cond}}(U_{\mathrm{Im}(f)}) \in \mathcal{D}(\mathrm{Cond\,Ab}).$$

Lemma 11.19. *Let $\mathfrak{X}$ be a quasiseparated semistable formal scheme over $\mathcal{O}_K$ and $\mathcal{U}$ the analytic covering of X given by a Zariski cover of $\mathfrak{X}$. Then*

$$\mathbf{R}\Gamma(\mathfrak{X}, \mathcal{O}_{\mathrm{cond}}) \simeq \check{C}(\mathcal{U}; \mathcal{O}^+_{\mathrm{cond}}).$$

This allows us to descend from the affine case (Theorem 11.14). The upshot is:

Theorem 11.20. *Let X be a smooth rigid analytic space over K and $\mathcal{U}$ a tamely semistable covering of X. Then there exists a natural morphism*

$$\alpha_{\mathcal{U}} : \check{C}(\mathcal{U}; \mathcal{O}^+_{\mathrm{cond}})[\varepsilon] \to \mathbf{R}\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}^+_X),$$

and each $H^i(\mathrm{cofib}(\alpha_{\mathcal{U}})$ is killed by some power of p depending only on $K, \mathcal{U}$ and i. If $\dim \mathcal{U} < \infty$, the power can be chosen uniformly in i.

Applying Lemma 11.10 again, we finally obtain Corollary 11.5.

12 Computing source of α and proof of Theorem B (19 Aug)

Today we complete the proof of [Bar+24, Theorem B]. Recall that the strategy from last time is to construct equivariant approximation maps

$$\alpha_{\mathrm{LT}} : A[\varepsilon] \to \mathbf{R}\Gamma(\mathrm{LT}_{K,\text{proét}}, \widehat{\mathcal{O}}^+_{\mathrm{cond}})$$
$$\alpha_{\mathcal{H}} : \mathbb{Z}_p[\varepsilon] \to \mathbf{R}\Gamma(\mathcal{H}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\mathrm{cond}})$$

such that the cofibre in $\mathcal{D}(\mathrm{Solid})$ satisfies

$$H^i(\mathrm{cofib}(\alpha)) = \begin{cases} 0 & i = 0 \\ p\text{-power torsion} & i \geq 1 \end{cases}$$

As a result, we obtain the diagram of solid complexes

$$A[\varepsilon]^{h\mathbb{G}_h} \xrightarrow{\alpha} \mathbf{R}\Gamma(\mathrm{LT}_K, \widehat{\mathcal{O}}^+_{\mathrm{cond}})^{h\mathbb{G}_h} \simeq \mathbf{R}\Gamma(\mathcal{H}, \widehat{\mathcal{O}}^+_{\mathrm{cond}})^{h\mathrm{GL}_h(\mathbb{Z}_p)} \xleftarrow{\alpha} \mathbb{Z}_p[\varepsilon]^{h\mathrm{GL}_h(\mathbb{Z}_p)} \tag{12.1}$$

in which all maps become equivalences when we invert p.

12.1 Integral cohomology

We begin with some refinements of Lazard's theorem to the integral case. Let G a p-adic Lie group with Lie algebra $\mathrm{Lie}(G)$, and $V/\mathbb{Q}_p$ be a finite dimensional G-representation. Then

$$H^*(G, V) \cong H^*(\mathrm{Lie}(G), V),$$

IM: I don't think we need this if we only care about the rational case?

and we saw in Talk 1 that when $G = \mathrm{GL}_h(\mathbb{Z}_p)$ or $\mathcal{O}_D^\times$ acting trivially on $\mathbb{Q}_p$ then this implies that

$$H^*(G, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, \ldots, x_{2h-1}).$$

Definition 12.2. A pro-p group U is *uniform* if

1. U is topologically finitely generated,
2. p is odd and U/U^p is abelian, or $p = 2$ and U/U^4 is abelian,
3. writing $U = U_1 \supset U_2 \supset \cdots$ for the lower p-central series, the index $[U_i : U_{i+1}]$ is independent of i.

For such a U, one can define an *integral* Lie algebra, a $\mathbb{Z}_p$-lattice $\mathcal{L}(U) \subset \mathrm{Lie}\, G$.

Theorem 12.3 (Hüber-Kings-Naumann)**.** *Let U be a uniform p-adic Lie group acting on a finitely generated free $\mathbb{Z}_p$-module M. Assume that $U \to \mathrm{Aut}(M)$ factors through $1 + p\,\mathrm{End}(M)$ if p is odd (or $1 + 4\,\mathrm{End}(M)$ when $p = 2$). Then $\mathcal{L}(U)$ acts on M, and*

$$H^*(U, M) \cong H^*(\mathcal{L}(U), M).$$

We now apply this in our case of interest:

Corollary 12.4. *Let $G = \mathrm{GL}_h(\mathbb{Z}_p)$ or $\mathcal{O}_D^\times$. Then:*

1. *For the trivial module $\mathbb{Z}_P$ we have*

$$H^i(G, \mathbb{Z}_p) = \mathbb{Z}_p^{\oplus r_i} \oplus S_i,$$

where $r_i = \dim_{\mathbb{Q}_p} \mathrm{gr}^i \Lambda_{\mathbb{Q}_p}(x_1, \ldots, x_{2h-1})$ and S_i is p-power torsion.

2. *Let C be a complex of solid $\mathbb{Z}_p$-modules with G-action. Suppose that $H^i(C) = 0$ for $i < 0$ and $H^*(C)$ is annihilated by a single power of p. Then $H^*(C^{hG})$ is annihilated by a single power of p.*

Proof. 1. In either case G is the group of units in a $\mathbb{Z}_p$ algebra A, and we set $U := 1 + p^2 A$—this is a uniform pro-p group. By Theorem 12.3 we know that

$$H^*(U, \mathbb{Z}_p) \cong H^*(\mathcal{L}(U), \mathbb{Z}_p),$$

and by Lazard's theorem

$$H^*(\mathcal{L}(U), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^*(\operatorname{Lie} U, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, \ldots, x_{2h-1}).$$

Thus

$$H^i(\mathcal{L}(U), \mathbb{Z}_p) \cong \mathbb{Z}_p^{\oplus r_i} \oplus S_i$$

by the classification of finitely generated $\mathbb{Z}_p$-modules. Also, cohomology of $\mathcal{L}(U)$ vanishes in degrees $i > n^2$, so there is a uniform bound for the torsion. To extend to G, use the maps

$$i^* : H^*(G, \mathbb{Z}_p) \rightleftarrows H^*(U, \mathbb{Z}_p) : i_!$$

having $i_! i^* = |G/U| = ??$.

Finish this

2. Use the Lyndon-Hochschild-Serre spectral sequence

$$H^i(U, \underline{H^j(C)}) \implies H^{i+j}(C^{hU}).$$

Since U has cohomological dimension n^2, if $p^r H^*(C) = 0$ then $p^{n^2 r} H^*(C^{hU}) = 0$. □

12.2 Proétale cohomlogy of the open ball

Recall the rigid analytic open ball

$$\mathring{\mathbb{D}}^d = \operatorname{Spa} \mathcal{O}_K[\![T_1, \ldots, T_d]\!] \setminus \{|p| = 0\},$$

which we studied in Talk 3. We saw there that this is not affinoid (the coordinate is not sheafy), but is nevertheless an adic space with affinoid covering

$$\mathring{\mathbb{D}}^d = \varinjlim_r \mathbb{D}^d_r,$$

where r varies over all $r = |p|^{m/n} < 1$. For such r,

$$\mathbb{D}^d_r := \left\{ x \in \mathring{\mathbb{D}}^d : |T(x)|^n \leq |p(x)|^m \right\}$$

is a rigid closed ball, which *is* affinoid. The goal of this section is to explain the computation of proétale cohomology

$$\mathbf{R}\Gamma(\mathring{\mathbb{D}}^d_{\text{proét}}, \widehat{\mathcal{O}}^+) \simeq A[\varepsilon], \tag{12.5}$$

since a priori the left-hand side is the domain of the map α_{LT}. To do this, we'll in fact refine to the covering by $\mathbb{D}^d_{r_\ell}$, where $r_\ell = |p|^{1/\ell}$ and $\ell \neq p$ is a prime. Note that $\mathbb{D}^d_{r_\ell}$ admits a smooth formal model after passing to the tamely ramified extension $L_\ell := K(p^{1/\ell})$: that is, $\mathbb{D}^d_{r_\ell}$ is the rigid generic fibre of the smooth formal scheme

$$\mathfrak{D}^d_{r_\ell} := \operatorname{Spf} \mathcal{O}_{L_\ell} \left\langle \frac{T_1}{p^{1/\ell}}, \ldots, \frac{T_d}{p^{1/\ell}} \right\rangle.$$

Last week, we saw:

Theorem 12.6. *Let X/K be a smooth rigid-analytic space of dimension d, and assume that $\mathcal{U} = \{U_i\}$ is a tamely semistable open cover. Then there is a natural map*

$$\alpha_{\mathcal{U}} : \check{C}(\mathcal{U}, \mathcal{O}^+_{\mathrm{cond}})[\varepsilon] \to \mathbf{R}\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}^+_{\mathrm{cond}})$$

with $H^i(\mathrm{cofib}(\alpha_{\mathcal{U}}))$ annihilated by some power of p. If the covering dimension of $\mathcal{U}$ is finite then we can pick a power that works uniformly for all i.

To apply this, note that for the tamely semistable covering $\mathcal{U} = \{\mathbb{D}^d_{r_\ell}\}$ we have

$$\check{C}(\mathcal{U}, \mathcal{O}^+_{\mathrm{cond}}) \simeq \mathbf{R}\varprojlim_{\ell} H^0(\mathbb{D}^d_{r_\ell}, \mathcal{O}^+_{\mathrm{cond}})$$

by definition. This limit can be nonzero only in degrees zero and one, and

$$\begin{aligned} H^0(\mathring{\mathbb{D}}^d, \widehat{\mathcal{O}}^+_{\mathrm{cond}}) &\cong \varprojlim_{\ell} H^0(\mathbb{D}^d_{r_\ell}, \mathcal{O}^+_{\mathrm{cond}}) \\ &\cong \varprojlim_{\ell} \mathcal{O}_K \left\langle \frac{T_1}{p^{1/\ell}}, \dots, \frac{T_1}{p^{1/\ell}} \right\rangle \\ &\cong \mathcal{O}_K[\![T_1, \dots, T_d]\!] \end{aligned}$$

In fact, [Bar+24, Lemma 6.2.3] shows that the $\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$-indexed system $\{H^0(\mathbb{D}^d_{r_\ell}, \mathcal{O}^+_{\mathrm{cond}})/p^j\}_{\ell,j}$ is Mittag-Leffler, so there is no H^1 and

$$\check{C}(\{\mathbb{D}^d_{r_\ell}\}, \mathcal{O}^+_{\mathrm{cond}}) \simeq \mathcal{O}_K[\![T_1, \dots, T_d]\!].$$

Combining this with Theorem 12.6 gives (12.5).

Corollary 12.7. *There is an isomorphism*

$$H^*(\mathbf{R}\Gamma(\mathrm{LT}_{K,\text{proét}}, \widehat{\mathcal{O}}^+_{\mathrm{cond}})^{h\mathcal{O}_D^\times}) \otimes_{W(\mathbb{F}_{p^h})} K \cong \Lambda_K(x_1, \dots, x_{2n-1})[\varepsilon] \oplus H^*(\mathcal{O}_D^\times, A^c) \otimes_{W(\mathbb{F}_{p^h})} K[\varepsilon].$$

In a similar way, one can compute the proétale cohomology of Drinfeld space. In this case $\mathcal{H}$ itself admits a semistable formal model $\mathfrak{H}/\mathbb{Z}_p$, a certain arrangement of projective spaces whose combinatorics was described by Drinfeld in terms of the Bruhat-Tits building for $\mathrm{PGL}_h(\mathbb{Q}_p)$.

Theorem 12.8. *There is a $\mathrm{GL}_h(\mathbb{Z}_p)$-equivariant equivalence*

$$\check{C}(\{\mathcal{H}_K\}, \mathcal{O}^+) \simeq \mathcal{O}_K.$$

Corollary 12.9. *There is an equivalence*

$$\mathbf{R}\Gamma(\mathrm{LT}_K, \widehat{\mathcal{O}}^+_{\mathrm{cond}})^{h\mathrm{GL}_h(\mathbb{Z}_p)} \otimes_{\mathcal{O}_K} K \simeq \Lambda_K(y_1, \dots, y_d)[\varepsilon].$$

In particular, using (12.1) we obtain an isomorphism

$$\Lambda_K(x_1, \dots, x_{2n-1})[\varepsilon] \oplus H^*(\mathcal{O}_D^\times, A^c) \otimes_{W(\mathbb{F}_p^h)} (\mathbb{F}_{p^h}) K[\varepsilon] \cong \Lambda_K(y_1, \dots, y_d)[\varepsilon],$$

so that $H^*(\mathcal{O}_D^\times, A^c) \otimes_{W(\mathbb{F}_p^h)} (\mathbb{F}_{p^h}) K[\varepsilon] = 0$ by counting dimensions. Using Lyndon-Hochschild-Serre for the inclusion $\mathcal{O}_D^\times \hookrightarrow \mathbb{G}_h$ we deduce that

$$H^*(\mathbb{G}_h, A^c) \otimes_{W(\mathbb{F}_{p^h})} K = 0,$$

which concludes the proof of the main theorem.

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